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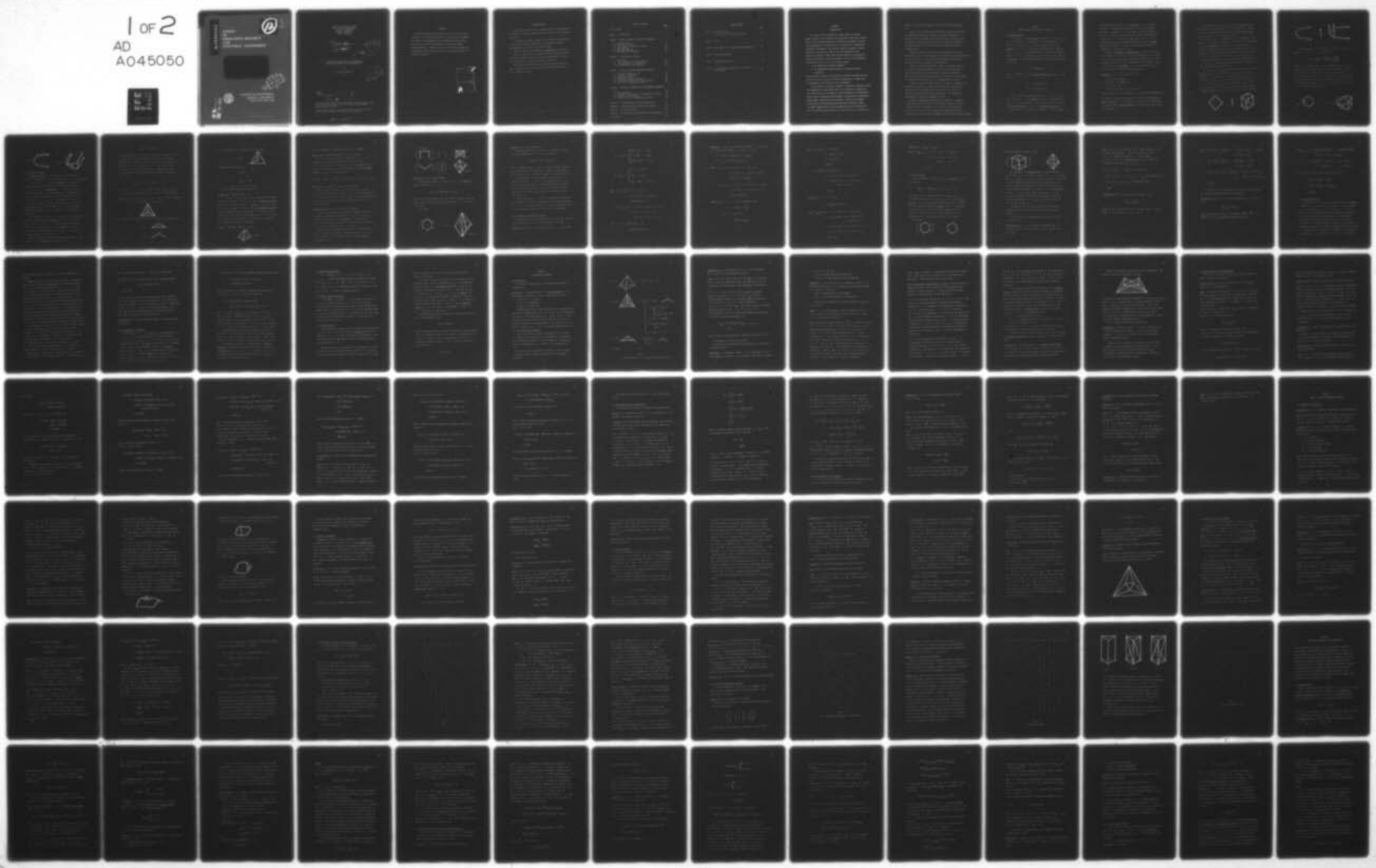

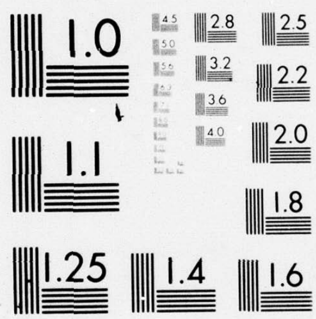

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9 TECHNICAL REP[redacted]T, NO. 354

12 130p.

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DECOMPOSITIONS, SHELLINGS, AND DIAMETERS OF SIMPLICIAL COMPLEXES AND CONVEX POLYHEDRA.

by

10 John Scott/Provan

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15 N00014-75-C-0678,
NSF-MPS-75-02034

This research was supported by the National Science Foundation under grants MPS75-02034 and by the Office of Naval Research under contract N00014-75-C-0678.

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ABSTRACT

Simplicial complexes are studied in their role dual to convex polyhedra, in particular, with respect to shellability and diameters. The concept of k-decomposability is presented which both insures shellability and limits the diameter of a simplicial complex. k-decomposability is studied in relation to standard techniques used in polyhedral theory, and with respect to topological properties of simplicial complexes.

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ACKNOWLEDGMENTS

Certainly the major part of my appreciation goes to Louis Billera, who advised me, funded me, and had almost a single hand in encouraging my research.

I am indebted to Richard Stanley for providing the initial motivation and several helpful suggestions, and to Marshall Cohen for straightening out some of the topological details.

A special thanks to Kathy King for a superlative job of typing.

Finally, I would like to congratulate the entire faculty of the School of Operations Research and Industrial Engineering at Cornell for providing such a healthy environment for research.

This research was supported by the National Science Foundation under grant MPS75-02024 and by the Office of Naval Research under contract N00014-75-C-0678.

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CHAPTER 1

INTRODUCTION

The class of convex polyhedra is closely linked to the more general classes of cell complexes and particularly simplicial complexes. Study of these complexes allows us to dispense with geometric and convexity arguments and concentrate more on topological or combinatorial properties of polyhedra. Many authors--Barnette, Brugesser, Danaraj, Klee, Mani, and Stanley, to name a few--have employed cell and simplicial complexes to shed light on such questions as shellability, diameter, and number of faces of polyhedra. In fact, two outstanding questions have arisen out of such studies, namely:

1) Are combinatorial spheres shellable?
2) Do combinatorial spheres satisfy the simplicial Hirsch conjecture?

For polytopes--a subclass of combinatorial spheres--the former is known to be true, and the truth of the latter corresponds to a twenty year old conjecture in the theory of linear programming.

In this thesis, we propose a property of simplicial complexes, called k-decomposability, which implies each of the properties in questions 1 and 2, and allows one more easily to bring topological and combinatorial facts to bear in studying these questions. The property of k-decomposability seems to behave nicely with respect to many of the standard tools used in studying polyhedra and simplicial complexes; indeed, the thesis suggests that vertex decomposability--the most restrictive form of k-decomposability--might be a property held by all combinatorial

spheres, a fact that would answer in the affirmative both questions 1 and 2.

Chapter 2 of the thesis includes the basic concepts in the theory of polyhedra and simplicial complexes, introducing shellability, diameters, Hirsch conjecture, and the basic tools used in their study. The concept of duality, linking polyhedra and simplicial complexes, is developed, and many of the ideas in these two theories are linked correspondingly.

Chapter 3 introduces the notion of k-decomposition and vertex decomposition in its strong and weak form, relating it to the concepts and tools in Chapter II. It is shown that k-decomposable complexes are shellable and have good bounds on their diameters.

Chapter 4 presents several major classes of complexes, some dual to polyhedra and some more general, which are vertex-decomposable. These include three classes not previously known to be shellable. The chapter concludes with some examples of complexes dual to polyhedra or combinatorial balls which are not k-decomposable for various k.

Chapter 5 studies k-decomposability as it relates to algebraic and piecewise linear topology. Necessary and sufficient conditions are established for k-decomposition and "partial k-decomposition" of homology and piecewise linear balls and spheres. Two interesting applications to shellings of spheres are cited.

Appendix 1 generalizes a result given in Chapter 3. Appendix 2 derives upper and lower bounds on the diameters of general complexes. Appendix 3 gives a restatement of the concept of k-decomposability in context of simple polyhedra. Appendix 4 relates the concept of shellability to the calculation of aggregate probabilities in Boolean systems.

CHAPTER 2

CONVEX POLYHEDRA AND SIMPLICIAL COMPLEXES

2.1 Convex Polyhedra

A set $A \subseteq \mathbb{R}^d$ is called <u>affine</u> if whenever $x,y \in A$ and $\lambda \in \mathbb{R}$ then $\lambda x + (1-\lambda)y \in A$. Equivalently, A is affine iff for $x \in A$, the set $A-x = \{y-x \mid y \in A\}$ is a linear subspace of $\mathbb{R}^d$. The <u>dimension</u> of A, $\dim A$, is the linear dimension of the unique set $A-x$, $x \in A$. If S is an arbitrary subset of $\mathbb{R}^d$ then the <u>affine hull</u> of S, aff S, is the unique smallest affine set containing S. A <u>hyperplane</u> is a $(d-1)$-dimensional affine space and can be described in the form

$$H = \{x \in \mathbb{R}^d \mid \langle \mu, x \rangle = \alpha\}$$

where $\mu \in \mathbb{R}^d \backslash \{0\}$, $\alpha \in \mathbb{R}$. The <u>closed half-spaces</u> defined by H are the sets

$$H^+ = \{x \in \mathbb{R}^d \mid \langle x, \mu \rangle \geq \alpha\}$$

$$H^- = \{x \in \mathbb{R}^d \mid \langle x, \mu \rangle \leq \alpha\}.$$

A set $C \subseteq \mathbb{R}^d$ is <u>convex</u> if whenever $x,y \in C$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)y \in C$. Its <u>dimension</u>, $\dim C$, is the dimension of aff C. If C is convex then C has an interior in aff C, called the <u>relative interior</u> of C, ri C. A <u>supporting hyperplane</u> to C is any hyperplane H in $\mathbb{R}^d$ which intersects C and one of whose

closed half spaces contains C. A proper face of C is any proper subset of C formed by the intersection of C with a supporting hyperplane. A face of C is any set which is either a proper face of C, C itself, or the null set; denote by $\mathcal{F}(C)$ the collection of faces of C. If C is convex, then every face of C is also convex and hence has a well-defined dimension. We call the convex hull of an arbitrary set $S \subseteq \mathbb{R}^d$ the unique smallest convex set containing S, and denote it conv S.

Finally, a polyhedron, or polyhedral set, is any set of the form $P = \{x \in \mathbb{R}^d | Ax \leq b\}$, where A is an $n \times d$ matrix and b an n-vector. P is a convex set, and we will assume that its dimension is d, for otherwise we can embed P by a linear map into $\mathbb{R}^k$, where $k = \dim P$. P is then called a d-polyhedron. If P is a polyhedron, then its face set $\mathcal{F}(P)$ is a finite collection satisfying:

Condition 2.1: Let F_1, F_2 be in $\mathcal{F}(P)$. Then

1) F_1, F_2 are polyhedra
2) $\mathcal{F}(F_i) = \{F \in \mathcal{F}(P) | F \subseteq F_i\}$
3) $F_1 \cap F_2 \in \mathcal{F}(P)$
4) If $F_1 \neq F_2$ then $\mathrm{ri}\, F_1 \cap \mathrm{ri}\, F_2 = \emptyset$

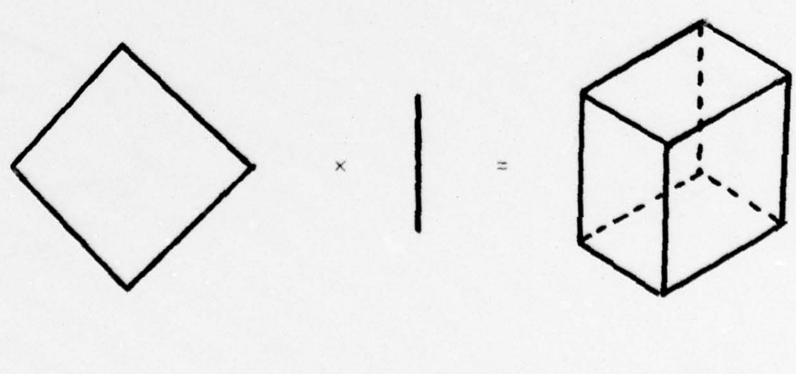
([17] Section 2.4, 11 and 12, section 2.6, 1 and 6, and [29] Theorem 18.3)

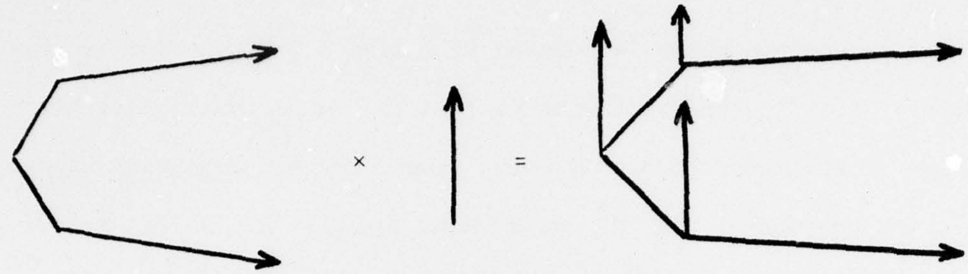
The 0-faces, 1-faces, and (d-1)-faces of P will be called vertices, edges, and facets of P respectively. Throughout this thesis we will be considering only pointed polyhedra, that is, polyhedra containing at least one vertex.

A polyhedron which is bounded is called a polytope and has the property ([17] p. 31) that each face is the convex hull of vertices in that face. Unbounded polyhedron always contain unbounded rays, that is, vectors β in $\mathbb{R}^d$ such that, for all x in P, $\lambda \geq 0$, $x + \lambda\beta$ is also in P. A d-polyhedron P is called simple if every vertex of P is contained in exactly d facets of P; a d-polytope P is called simplicial if each facet of P contains exactly d vertices. A simple polyhedron has the convenient additional property ([17] p. 65, problem 12) that every k-dimensional face of P is the intersection of exactly $d-k$ facets of P, and conversely, that any non-empty intersection of $d-k$ facets forms a k-dimensional face.

Two special constructions should be noted here, as appeared in [23], namely the product and the wedge.

If P_1, P_2 are d_1- and d_2-polyhedra, respectively, then their product $P_1 \times P_2 = \{(x,y) \in \mathbb{R}^{d_1+d_2} | x \in P_1,\ y \in P_2\}$ is a (d_1+d_2)-polyhedron. Its k-faces are all products $F_1 \times F_2$ where F_1 is a p-face of P_1 and F_2 is a k-p face of P_2, $k = 0,\ldots,d_1+d_2$, $0 \leq p \leq k$. If P_1, P_2 are polytopes, then so is $P_1 \times P_2$, and if P_1, P_2 are simple then so is $P_1 \times P_2$. Two special cases, the open and closed prisms, occur when $P_2 = [0,1]$ and $[0,\infty)$ respectively.

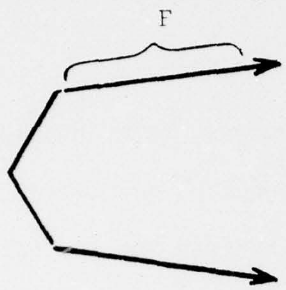


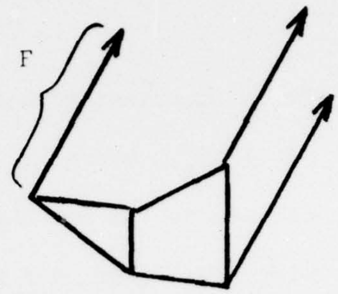


Given d-polyhedron $P = \{x \in \mathbb{R}^d | Ax \leq b\}$, facet F of P, the <u>wedge</u> of P with foot F is the polyhedron

$$w(P,F) = \left\{ x \in \mathbb{R}^{d+1} \middle| \begin{pmatrix} A & 0 \\ 0 & -1 \\ \mu & 1 \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \\ \alpha \end{pmatrix} \right\}$$

where $H = \{x \in \mathbb{R}^d | \langle \mu, x \rangle = \alpha\}$ is the hyperplane defining F, $P \subseteq H^-$. Geometrically, it can be described as a closed pyramid on P with the top and bottom facets, $P \times \{0\}$ and $P \times \{1\}$, identified at F. A face of $w(P,F)$ is either a face of one of the top or bottom facets or a closed pyramid on a face E of P with $E \cap F \times \{0\}$, $E \cap F \times \{1\}$ (if they exist) identified. If P is a polytope, then so is $w(P,F)$; if P is simple, then so is $w(P,F)$.

$(P,F) =$ 

$\Rightarrow w(P,F) =$ 

2.2 Simplicial Complexes

A simplicial complex Σ on a finite set E is a non-empty collection of subsets of E with the property that $\sigma \in \Sigma$, $\tau \subseteq \sigma$ implies $\tau \in \Sigma$. An element $\sigma \in \Sigma$ is called a simplex in (or face of) Σ, and will be identified, when necessary, by listing its vertices $\sigma = v_1 \ldots v_n$, $v_i \in E$. The dimension of σ is $\dim \sigma = |\sigma|-1$; the simplices of dimension 0 are called vertices, and are denoted as a set $V(\Sigma)$. (Note: though every vertex is an element of E, not every element of E is a vertex). The dimension of Σ, $\dim \Sigma$, is equal to the dimension of the largest simplex and Σ is called pure dimensional if all maximal simplices in Σ are of the same dimension. The size of Σ, $|\Sigma|$, is equal to the total number of simplices in Σ.

For an arbitrary collection Σ of subsets of E, we define the simplicial complex called the closure of Σ, $\mathrm{cl}\,\Sigma$ or $\overline{\Sigma}$, to be $\mathrm{cl}\,\Sigma = \{\tau \mid \tau \subseteq \sigma$ for some $\sigma \in \Sigma\}$. We will extend the notation to allow $\mathrm{cl}\,\sigma = \overline{\sigma} = \mathrm{cl}\{\sigma\}$ for $\sigma \in \Sigma$. The boundary of Σ, $\partial\Sigma$, is the closure of the collection of $(d-1)$-simplices of Σ which are contained in exactly one d-simplex of Σ.

We define finally the key concepts of deletion and link. Given simplex $\sigma \in \Sigma$ the deletion of σ from Σ is

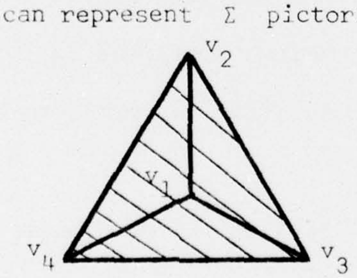
$$\Sigma \backslash \sigma = \{\tau \in \Sigma \mid \sigma \not\subseteq \tau\}.$$

It is important to note the difference between this notation and that of $\Sigma_1 \backslash \Sigma_2$, where Σ_2 is a simplicial complex, the latter being defined as the set theoretic difference between the two collections of simplices. For $\sigma \neq \emptyset$ $(\Sigma \backslash \emptyset = \emptyset)$, $\Sigma \backslash \sigma$ is a simplicial complex. For multiple deletions we will simply write $\Sigma \backslash \sigma_1 \backslash \ldots \backslash \sigma_n$ and take it to mean deletions are performed from left to right. The link of σ in Σ is

$$\ell k_{\Sigma} \sigma = \{\tau \in \Sigma \mid \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Sigma\}.$$

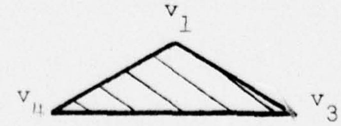
For every σ in Σ, $\ell k_{\Sigma} \sigma$ is a simplicial complex $(\ell k_{\Sigma} \emptyset = \Sigma)$, and if Σ is pure d-dimensional then $\ell k_{\Sigma} \sigma$ is pure $(d-|\sigma|)$-dimensional.

Example: If $\Sigma = cl\{v_1v_2v_3, v_1v_2v_4, v_1v_3v_4\}$, then Σ is pure of dimension 2, and $|\Sigma| = 3$. We can represent Σ pictorially as

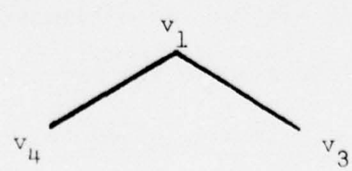


Some examples of deletions and links, also represented pictorially are:

$\Sigma \backslash v_2 =$

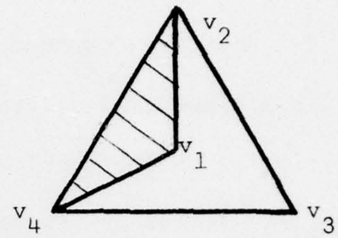


$\ell k_{\Sigma} v_2 =$



both pure 2-dimensional and 1-dimensional, respectively,

$\Sigma \backslash v_1 v_3 =$



2-dimensional but not pure, and

$$\ell k_\Sigma v_1 v_3 = \begin{array}{l} \quad v_2. \\ v_4. \end{array}$$

$$\ell k_\Sigma v_1 v_2 v_3 = \{\emptyset\}$$

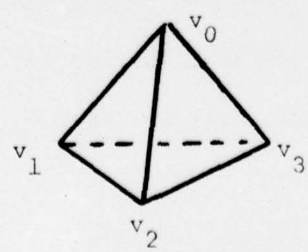
$\{\emptyset\}$ is a (-1)-dimensional simplicial complex.

Several constructions should be noted here:

<u>Example 2.2.1</u>: Simplexes ([17] p. 53).

If $v = \{v_0, \ldots, v_d\}$ is a set of $d+1$ points in $\mathbb{R}^d$ for which $\dim(\text{conv } V) = d$, then $T^d = \text{conv } V$ is called a <u>geometric</u> <u>d-simplex</u>. Every T^d has $F(T^d) = \{\text{conv}\{v_{i_1}, \ldots, v_{i_k}\} | \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V\}$. Hence the simplicial complex $\Delta^d = \text{cl}\{V\}$, called a <u>combinatorial</u> <u>d-simplex</u> (or just d-simplex) has the property that Δ^d is realized as the collection of those vertex sets which define a face of T^d. We allow, for completeness, the combinatorial (-1)-simplex $\Delta^{-1} = \{\emptyset\}$ which has no corresponding geometric realization.

<u>Example</u>: A geometric 3-simplex looks like



(solid)

and its corresponding combinatorial simplex is $\Sigma = \overline{v_0v_1v_2v_3}$.

Example 2.2.2: Boundary Complexes of Simplicial Polytopes.

If P is a simplicial polytope, then each facet F of P is geometric d-simplex on the vertices of F, so that the set $\Sigma_P = \{v_1\ldots v_k | v_i$ vertex of P, $\text{conv}\{v_1,\ldots,v_k\}$ is a proper face of $P\}$ forms a (d-1)-dimensional pure simplicial complex, called the boundary complex of P. Σ_{T^d} is one example, with

$$\Sigma_{T^d} = \partial\Delta^d = \Delta^d \setminus v_0\ldots v_d = \text{cl}\{v_0\ldots v_{d-1}, v_0\ldots v_{d-2}v_d, \ldots, v_0v_2\ldots v_d, v_1\ldots v_d\}.$$

Example 2.2.3: The Union of Two Simplicial Complexes.

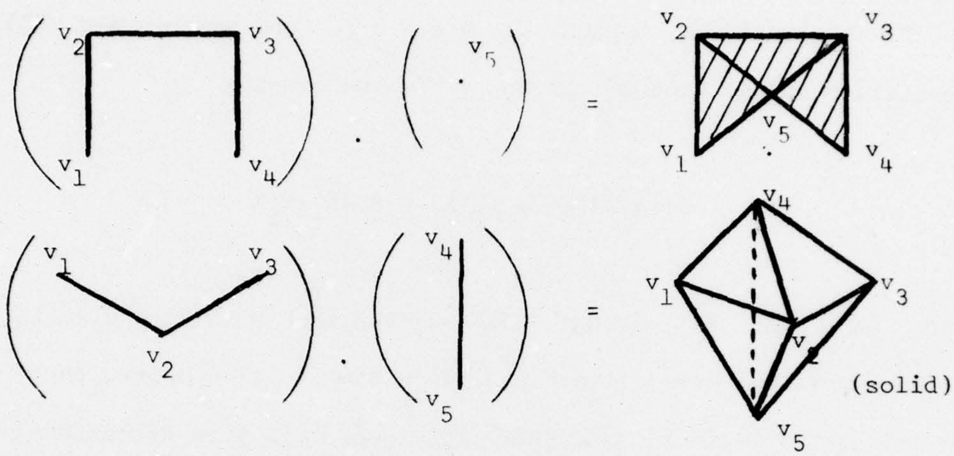
Given simplicial complexes Σ_1,Σ_2 on vertex sets $V(\Sigma_1),V(\Sigma_2)$ (not necessarily disjoint) their union $\Sigma_1 \cup \Sigma_2$ (in the regular sense) is also a simplicial complex. We have $\dim(\Sigma_1 \cup \Sigma_2) = \max\{\dim \Sigma_1, \dim \Sigma_2\}$, and Σ_1 and Σ_2 being pure d-dimensional implies that $\Sigma_1 \cup \Sigma_2$ is pure d-dimensional.

Example 2.2.4: The Join of Two Simplicial Complexes.

Given Σ_1,Σ_2 two simplicial complexes on disjoint vertex sets $V(\Sigma_1),V(\Sigma_2)$, their join is the complex $\Sigma_1.\Sigma_2 = \{\sigma_1 \cup \sigma_2 | \sigma_i \in \Sigma_i\}$ on vertex set $V(\Sigma_1) \cup V(\Sigma_2)$ with factors Σ_1 and Σ_2. Then $\Sigma_1.\Sigma_2$ is also a simplicial complex and $\dim(\Sigma_1+\Sigma_2) = \dim \Sigma_1 + \dim \Sigma_2$, and Σ_1 and Σ_2 being pure implies $\Sigma_1.\Sigma_2$ pure.

Again, we will abuse the notation to include collections which are not complexes and simplices in complexes (such as $\sigma.\Sigma_2$ $\sigma \in \Sigma_1$). In the case where Σ_2 is a point set we call $\Sigma_1.\Sigma_2$ a multiple suspension (on Σ_1).

For instance,



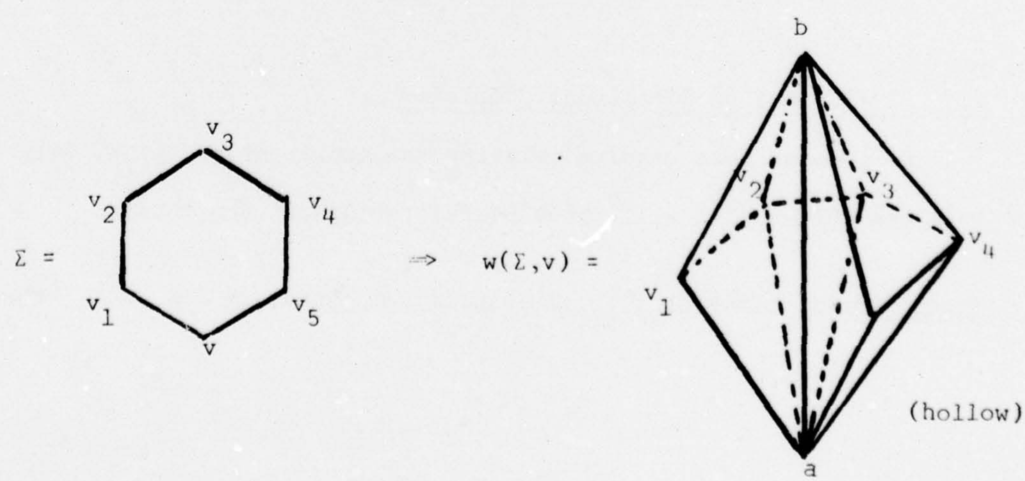
Example 2.2.5: The Simplicial Wedge.

Given simplicial complex Σ, vertex v in Σ, the simplicial wedge (or wedge) of Σ on v is

$$w(\Sigma,v) = \{a,b,\emptyset\}.(\Sigma\backslash v) \cup ab.\ell k_{\Sigma} v$$

where a,b are two additional vertices. $w(\Sigma,v)$ is again a simplicial complex and its dimension is one more than $\dim \Sigma$. If Σ is pure then so is $w(\Sigma,v)$.

For instance



Example 2.2.6: Stellar Subdivision.

Given simplicial complex Σ, $\emptyset \neq X \in \Sigma$, and symbol $a \notin V(\Sigma)$, the stellar subdivision of Σ on τ is the complex

$$st(a,X)[\Sigma] = (\Sigma \setminus X) \cup a.\partial X.\ell k_{\Sigma} X$$

where $\partial X = \overline{X} \setminus X$. Noting that $(\Sigma \setminus X) \cup a.\partial X.\ell k_{\Sigma} X = (\Sigma \setminus X) \cup \overline{a}.\partial X.\ell k_{\Sigma} X$, it is easy to see that $st(a,X)[\Sigma]$ is a simplicial complex, that $\dim st(a,X)[\Sigma] = \dim \Sigma$, and that $st(a,X)[\Sigma]$ is pure dimensional if Σ is pure dimensional. Generally we will call a complex Σ_1 a stellar subdivision of Σ_2 if Σ_1 can be obtained from Σ_2 by a series of stellar subdivisions. Stellar subdivisions are important because they comprise the building blocks of piecewise linear (or combinatorial) complexes and in particular of simplicial polytopes (see chapter 5).

One interesting note: If $w(\Sigma,v)$ is the simplicial wedge with extra vertices a,b, then $st(v,ab)[w(\Sigma,v)]$ is the double suspension of Σ on a,b. The wedge can then, in a sense, be obtained by taking a double suspension followed by an "inverse stellar subdivision."

2.3 Five Lemmas on Simplicial Complexes

We present here results relating the constructions link, deletion, join and union which will prove useful throughout the thesis.

Lemma 2.3.1: Given Σ_1, Σ_2 simplicial complexes, $\tau \in \Sigma_1 \cup \Sigma_2$ then

$$\text{i)}\quad \Sigma_1 \cup \Sigma_2 \backslash \tau = \begin{cases} (\Sigma_1 \backslash \tau) \cup \Sigma_2 & \tau \in \Sigma_1 \backslash \Sigma_2 \\ \Sigma_1 \cup (\Sigma_2 \backslash \tau) & \tau \in \Sigma_2 \backslash \Sigma_1 \\ (\Sigma_1 \backslash \tau) \cup (\Sigma_2 \backslash \tau) & \tau \in \Sigma_1 \cap \Sigma_2 \end{cases}$$

$$\text{ii)}\quad \ell k_{\Sigma_1 \cup \Sigma_2} \tau = \begin{cases} \ell k_{\Sigma_1} \tau & \tau \in \Sigma_1 \backslash \Sigma_2 \\ \ell k_{\Sigma_2} \tau & \tau \in \Sigma_2 \backslash \Sigma_1 \\ \ell k_{\Sigma_1} \tau \cup \ell k_{\Sigma_2} \tau & \tau \in \Sigma_1 \cap \Sigma_2 \end{cases}$$

Proof: i) $\Sigma_1 \cup \Sigma_2 \backslash \tau = \{\sigma \in \Sigma_1 \cup \Sigma_2 | \tau \not\leq \sigma\}$

$= \{\sigma \in \Sigma \, | \tau \not\leq \sigma\} \cup \{\sigma \in \Sigma_2 | \tau \not\leq \sigma\}$

$=$ right hand side of i.

ii) $\ell k_{\Sigma_1 \cup \Sigma_2} \tau = \{\sigma \in \Sigma_1 \cup \Sigma_2 | \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Sigma_1 \cup \Sigma_2\}$

$= \{\sigma \in \Sigma_1 | \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Sigma_1\}$

$\cup \{\sigma \in \Sigma_2 | \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in \Sigma_2\}$,

since $\tau \cup \sigma \in \Sigma_i$ implies that $\sigma \in \Sigma_i$,

$=$ right hand side of ii.

Lemma 2.3.2: Given Σ_1, Σ_2 simplicial complexes, $\tau \in \Sigma_1 . \Sigma_2$. Let τ_{Σ_i} be the set of vertices in τ and in Σ_i. Then

i) $(\Sigma_1 . \Sigma_2) \backslash \tau = (\Sigma_1 \backslash \tau_{\Sigma_1}) . \Sigma_2 \cup \Sigma_1 . (\Sigma_2 \backslash \tau_{\Sigma_2})$

ii) $\ell k_{\Sigma_1 . \Sigma_2} \tau = (\ell k_{\Sigma_1} \tau_{\Sigma_1}) . (\ell k_{\Sigma_2} \tau_{\Sigma_2})$.

Proof: i) $\Sigma_1 . \Sigma_2 \backslash \tau = \{\sigma_1 \cup \sigma_2 | \sigma_i \in \Sigma_i \quad \tau \not\subseteq \sigma_1 \cup \sigma_2\}$

$$= \{\sigma_1 \cup \sigma_2 | \sigma_i \in \Sigma_i, \quad \tau_{\Sigma_1} \not\subseteq \sigma_1, \tau_{\Sigma_2} \not\subseteq \sigma_2\}$$

$$= (\Sigma_1 \backslash \tau_{\Sigma_1}) . \Sigma_2 \cup \Sigma_1 . (\Sigma_2 \backslash \tau_{\Sigma_2})$$

ii) $\ell k_{\Sigma_1 . \Sigma_2} \tau = \{\sigma_1 \cup \sigma_2 | \sigma_i \cap \tau = \emptyset, \sigma_1 \cup \sigma_2 \cup \tau \in \Sigma_1 . \Sigma_2\}$

$$= \{\sigma_1 \cup \sigma_2 | \sigma_i \cap \tau_{\Sigma_i} = \emptyset, \quad \sigma_i \cup \tau_i \in \Sigma_i\}$$

$$= (\ell k_{\Sigma_1} \tau_{\Sigma_1}) . (\ell k_{\Sigma_2} \tau_{\Sigma_2}).$$

Lemma 2.3.3: i) If σ, τ are non-comparable, then

$$(\Sigma \backslash \sigma) \backslash \tau = (\Sigma \backslash \tau) \backslash \sigma$$

ii) If $\tau \in \ell k_\Sigma \sigma$ then

$$(\ell k_\Sigma \sigma) \backslash \tau = \ell k_{\Sigma \backslash \tau} \sigma$$

Proof: i) $(\Sigma\backslash\sigma)\backslash\tau = \{\nu \in \Sigma\backslash\sigma | \tau \not\leq \nu\}$

$$= \{\nu \in \Sigma | \sigma \not\leq \nu,\ \tau \not\leq \nu\}$$

$$= \{\nu \in \Sigma\backslash\tau | \sigma \not\leq \nu\}$$

$$= (\Sigma\backslash\tau)\backslash\sigma$$

ii) $(\ell k_\Sigma \sigma)\backslash\tau = \{\nu \in \ell k_\Sigma \sigma | \tau \not\leq \nu\}$

$$= \{\nu \in \Sigma | \nu \cap \sigma = \emptyset,\ \nu \cup \sigma \in \Sigma,\ \tau \not\leq \nu\}$$

$$= \{\nu \in \Sigma\backslash\tau | \nu \cap \sigma = \emptyset,\ \nu \cup \sigma \in \Sigma,\ \tau \not\leq \sigma \cup \nu\}$$

(since $\tau \cap \sigma = \emptyset$)

$$= \ell k_{\Sigma\backslash\tau}\sigma$$

Lemma 2.3.4: If $\sigma \cup \tau \in \Sigma$ then

$$\ell k_{\ell k_\Sigma \sigma}(\tau\backslash\sigma) = \ell k_\Sigma(\tau \cup \sigma)$$

Proof: $\ell k_{\ell k_\Sigma \sigma}(\tau\backslash\sigma) = \{\nu \in \Sigma | \nu \cap (\tau\backslash\sigma) = \emptyset,\ \nu \cup (\tau\backslash\sigma) \in \ell k_\Sigma \sigma\}$

$$= \{\nu \in \Sigma | \nu \cap (\tau\backslash\sigma) = \emptyset,\ [\nu \cup (\tau\backslash\sigma)] \cap \sigma = \emptyset,\ \nu \cup (\tau\backslash\sigma) \cup \sigma \in \Sigma\}$$

$$= \{\nu \in \Sigma | \nu \cap (\tau\backslash\sigma) = \emptyset,\ \nu \cap \sigma = \emptyset,\ \nu \cup \tau \cup \sigma \in \Sigma\}$$

$$= \{\nu \in \Sigma | \nu \cap (\tau \cup \sigma) = \emptyset,\ \nu \cup (\tau \cup \sigma) \in \Sigma\}$$

$$= \ell k_\Sigma(\tau \cup \sigma).$$

Lemma 2.3.5: $\ell k_{\Sigma\setminus(\sigma\setminus\tau)}\tau = \ell k_{\Sigma\setminus\sigma}\tau$

Proof: $\ell k_{\Sigma\setminus(\sigma\setminus\tau)}\tau = \{\nu \in \Sigma \mid \nu \cap \tau = \emptyset,\ \nu \cup \tau \in \Sigma,\ \sigma\setminus\tau \nsubseteq \nu\}$

$= \{\nu \in \Sigma \mid \nu \cap \tau = \emptyset,\ \nu \cup \tau \in \Sigma,\ \sigma \nsubseteq \nu$

(since $\nu \cap \sigma = \emptyset$)

$= \ell k_{\Sigma\setminus\sigma}\tau$

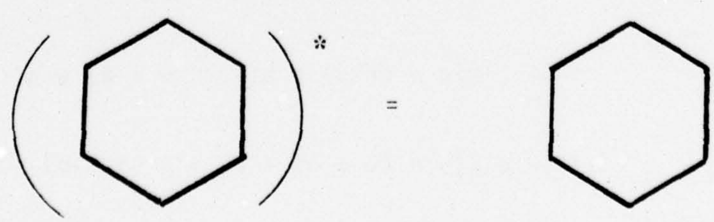
2.4 Dual Complexes

Given d-polyhedron $P \subseteq \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, the polar set to P at x_0 is defined

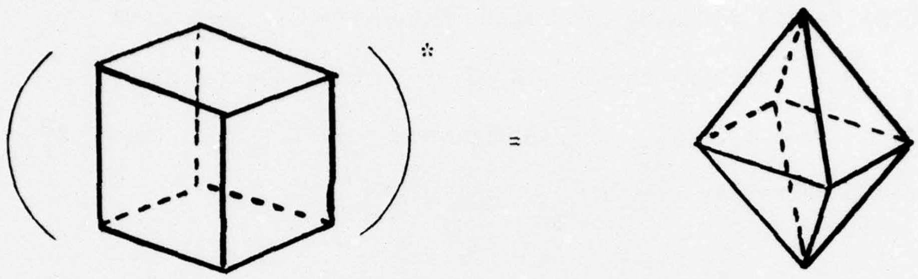
$$P^*_{x_0} = \{y \in \mathbb{R}^d \mid \langle y-x_0, z\rangle \leq 1 \text{ for each } z \in P\}.$$

Then $P^*_{x_0}$ is also a polyhedron ([17] p. 49, prob. 5 viii). If P is a polytope, and $x_0 \in \operatorname{int} P$, then $P^*_{x_0} = P^*$ is called a dual polytope to P and has the property that there exists an inclusion reversing isomorphism between the proper faces of P and the faces of P^* ([17] p. 47). Hence, among other properties, P^* is unique up to $F(P)$, $P^{**} = P^*$, P simple implies that P^* is simplicial, and P simplicial implies that P^* is simple.

For instance,



(All 2-polytopes are self-dual.) And



More generally, the <u>simplicial</u> <u>dual</u> to a simple polyhedron P is the complex Σ_P^* on $E = \{v_i | f_i$ a facet of P $i = 1 \dots n\}$ defined $\Sigma_P^* = \{v_{i_1} \dots v_{i_k} | f_{i_1} \cap \dots \cap f_{i_k}$ is a non-null face of $P\}$. Then Σ_P^* is a (d-1)-dimensional simplicial complex, and again there is an inclusion reversing isomorphism ϕ_P between the proper focus of P and the non-empty simplices of Σ_P where if F is the intersection of facets $f_{i_1}, \dots, f_{i_k}$ then $\phi_P(F)$ is the simplex $v_{i_1} \dots v_{i_k}$. To relate the simplicial dual to the dual polytope, we simply note that for simple polytope P, $\Sigma_P^* = \Sigma_{P^*}$, the boundary complex of the simplicial polytope P^*.

One immediate result is that any property true of general simplicial complexes is true in its dual sense for all simple polyhedra.

We now relate some of the polyhedra in section 2.1 to their simplicial duals.

<u>Proposition 2.4.1</u>: Let P be a simple d-polyhedron, Σ_P^* its simplicial dual, F a proper face of P, and $\sigma = \phi_P(F)$. Then $\Sigma_F^* = \ell k_{\Sigma_P^*} \sigma$.

Proof: Let $f_1 \ldots f_n$ be the facets of P, chosen so that $f_1, \ldots, f_k$ are those facets defining F. Then $V(\Sigma) = \{v_1, \ldots, v_n\}$ and $\sigma = v_1 \ldots v_k$. Further, the facets of F are of the form $\overline{f}_i = f_1 \cap \ldots \cap f_k \cap f_{j_i}$ for appropriate $j_1, \ldots, j_m > k$. Then Σ_F^* is defined on vertex set $\{v_{j_1}, \ldots, v_{j_m}\}$ by

$$\Sigma_F^* = \{v_{i_1} \ldots v_{i_\ell} \mid \overline{f}_{i_1} \cap \ldots \cap \overline{f}_{i_\ell} \neq \emptyset \text{ is a face of } F\}$$

$$= \{v_{i_1} \ldots v_{i_\ell} \mid f_1 \cap \ldots \cap f_k \cap f_{j_{i_1}} \cap \ldots \cap f_{j_{i_\ell}} \neq \emptyset \text{ is a face of } P\}$$

$$= \{v_{i_1} \ldots v_{i_\ell} \mid v_1 \ldots v_k v_{i_1} \ldots v_{i_\ell} \text{ is a simplex of } \Sigma_P^*\}$$

$$= \ell k_{\Sigma_P^*} \sigma .$$

Proposition 2.4.2: Let P_1, P_2 be simple polyhedra. Then

$$\Sigma^*_{P_1 \times P_2} = \Sigma^*_{P_1} . \Sigma^*_{P_2} .$$

Proof: We have the facets of $P_1 \times P_2$ of the form $f_j^1 \times P_2$ or $P_1 \times f_j^2$, where f_j^i is a facet of P_i $j = 1, \ldots, n_i$. So

$$\Sigma^*_{P_1\times P_2} = \{v^1_{i_1}\ldots v^1_{i_k} v^2_{j_1}\ldots v^2_{j_\ell} \mid f^1_{i_1}\times P_2 \cap \ldots \cap f^1_{i_k}\times P_2 \cap P_1\times f^2_{j_1} \cap \ldots \cap P_1\times f^2_{j_\ell} \text{ is a face of } P_1\times P_2\}$$

$$= \{v^1_{i_1}\ldots v^1_{i_k} v^2_{j}\ldots v^2_{j_\ell} \mid (f^1_{i_1} \cap \ldots \cap f^1_{i_k})\times(f^2_{j_1} \cap \ldots \cap f^2_{j_\ell}) \text{ if a face of } P_1\times P_2\}$$

$$= \{\{v^1_{i_1},\ldots,v^1_{i_k}\} \cup \{v^2_{j_1},\ldots,v^2_{j_\ell}\} \mid f^1_{i_1} \cap \ldots \cap f^1_{i_k} \text{ is a face of } P_1, f^2_{j_1} \cap \ldots \cap f^2_{j_\ell} \text{ is a face of } P_2\}$$

$$= \Sigma^*_{P_1}\cdot\Sigma^*_{P_2}.$$

As special cases, the dual complexes to the open and closed prisms are the single and double suspensions, respectively.

Proposition 2.4.3: Let P be a simple polyhedron, f_{i_0} a facet of P. Then

$$\Sigma^*_{w(P,f_{i_0})} = w(\Sigma^*_P, v_{i_0}).$$

Proof: The facets of $w(P,f_{i_0})$ can be denoted $\overline{P\times\{0\}}$, $\overline{P\times\{1\}}$, and $\overline{f_i\times[0,1]}$ $i \neq i_0$ (with the proper identifications), and

$$\Sigma^*_{w(P,f_{i_0})} = \{v^1_{i_1}\cdots v^1_{i_k}(v^2_0)(v^2_1) \mid \overline{f_{i_1}\times[0,1]} \cap \cdots \cap \overline{f_{i_k}\times[0,1]}(\cap\ \overline{P\times\{0\}})$$

$$(\cap\ \overline{P\times\{1\}}) \text{ is a face of } w(P,f_{i_0})\}$$

$$= \{v^1_{i_1}\cdots v^1_{i_k}(v^2_i) \mid (i=0,1)\ f_{i_1} \cap \cdots \cap f_{i_k} \text{ a face of } P\}$$

$$\cup\ \{v^1_{i_1}\cdots v^1_{i_k} v^2_0 v^2_1 \mid f_{i_1} \cap \cdots \cap f_{i_k} \text{ a face of } F\},$$

since the top and bottom faces intersect exactly at F,

$$= \{v^2_0, v^2_1, \emptyset\}.(\Sigma^*_P \backslash v_{i_0}) \cup v^2_0 v^2_1.\Sigma^*_F$$

$$= \{v^2_0, v^2_1, \emptyset\}.(\Sigma^*_P \backslash v_{i_0}) \cup v^2_0 v^2_1.\ell k_{\Sigma^*_P} v_{i_0}$$

$$= w(\Sigma^*_P, v_{i_0}).$$

2.5 Shellable Complexes

A collection $\mathcal{U}$ of k-faces in a polyhedral complex is <u>shellable</u> if either $k = 0$ (a point set) or the k-faces in $\mathcal{U}$ can be ordered $f_1,\ldots,f_n$ so that, for $j = 2,\ldots,n$, $f_j \cap (f_{j-1} \cup \cdots \cup f_1)$ is a union of $(k-1)$-faces which themselves form a shellable collection. In a simplicial polytope the restriction that the collection of $(k-1)$-faces is shellable can be dropped, since any collection of $(k-1)$-faces in a geometric k-simplex is shellable (in any order). The corresponding definition for simplicial complexes, then, can be stated:

A pure d-dimensional complex Σ is shellable if its d-simplices can be ordered $\sigma_1,\ldots,\sigma_n$ so that, for $j = 2,\ldots,n$, $\bar{\sigma}_j \cap (\bigcup_{i=1}^{j-1} \bar{\sigma}_j)$

(the intersection taken as complexes) is a pure (d-1)-dimensional complex.

Brungesser and Mani [8] showed that all polytopes are shellable. Their shelling is called a line shelling, and is produced as follows. Let H_i be the hyperplane defining facet f_i, $i = 1,\ldots,n$ and let ℓ be any line in R^d passing through $\text{int}(P)$ with the properties 1) ℓ intersects every H_i and 2) ℓ intersects no more than one H_i at each of its points. Then the order of shelling of the f_i is the order in which the H_i are intersected by a path beginning at some point x in $\ell \cap \text{int } P$ travelling out from x along one ray of ℓ, then back in to x along the opposite ray. In the same paper, they showed that shellability of simplicial complexes is preserved under stellar subdivisions, and as a result that every simplicial complex which can be "realized" as a combinatorial sphere or ball (see Chapter 5) has a series of stellar subdivisions which produces a shellable complex.

One can extend polytope shelling to a shelling of the dual complex of a simple unbounded polyhedron P as follows: first choose a hyperplane $H = \{x \langle \mu,x\rangle = \alpha\}$ which supports some vertex v of P, $P \subseteq H^-$. Let $\alpha_* = \min\{\langle\mu,v\rangle | v$ a vertex of $P\}$ (which exists since there are only a finite number of vertices in P). Let $H_1 = \{x | \langle\mu,x\rangle = \alpha_* - 1\}$, and consider the polyhedron $P_1 = P \cap H_1^+$. P_1 is a polytope, since any unbounded ray of P_1 (and thus of P) must be parallel to H (in order that every $x \in P_1$ have $\alpha \leq \langle\mu,x\rangle \leq \alpha_* - 1$), implying that H does not support P uniquely at v. The vertices of P_1 are exactly those of P along with $H_1 \cap L$,

where L is an unbounded edge of P. Hence P_1 is simple, and

$$\Sigma^*_{P_1} = \Sigma^*_P \cup \{v \cup \sigma \mid \sigma \text{ is a (d-2)-simplex in } \Sigma^*_P \text{ contained in exactly one (d-1)-simplex of } \Sigma^*_P\}$$

$$= \Sigma^*_P \cup v.\partial\Sigma^*_P.$$

Let x_0 be the vertex of the simplicial polytope P^*_1 corresponding to v. We can always choose a shelling line ℓ to P^*_1 which crosses the facets containing v before any other, by choosing ℓ sufficiently close to x_0, with ℓ normal to a supporting hyperplane to x_0. Hence the line shelling described removes the additional facets of $\Sigma^*_{P_1}$ first, and then shells Σ^*_P. We have, then

Proposition 2.5.1: The dual simplicial complex to a simple polyhedron is shellable.

2.6 The Diameter of Complexes

Given vertices v_0, v_1 in a polyhedron P_1 an edge path from v_0 to v_1 in P is an alternating sequence of vertices and edges $v_0 = u_0, e_1, u_1, \ldots, u_{k-1}, e_k, u_k = v_1$ of P such that u_{i-1}, u_i are (the) incident vertices to e_i. The length of such a path is k = the number of edges = one less than the number of vertices. The distance $\ell(v_0, v_1)$ between v_0 and v_1 is the minimum length of an edge path between v_0 and v_1, and the diameter of P, diam P, is the maximum of $\ell(v_0, v_1)$ as v_0, v_1 range over all vertices of P. We have P path connected [17, §11.3], and so diam $P < \infty$.

We note here that of the two polyhedral constructions in Section 1,

1) $\operatorname{diam} P_1 \times P_2 = \operatorname{diam} P_1 + \operatorname{diam} P_2$, and

2) $\operatorname{diam}(w(P,F)) \geq \operatorname{diam} P$.

The first equality can be seen by observing that any two vertices $\nu_1 \times \nu_2$, $\mu_1 \times \mu_2$ of $P_1 \times P_2$ can be joined by edge path

$$\nu^1 \times \nu^2 = u_0^1 \times u_0^2,\ e_1^1 \times u_0^2, \ldots, e_{n_1}^1 \times u_0^2,\ u_{n_1}^1 \times u_0^2,$$

$$u_{n_1}^1 \times e_1^2,\ u_{n_1}^1 \times v_1^2, \ldots, u_{n_1}^1 \times e_{n_2}^2,\ u_{n_1}^1 \times u_{n_2}^2 = \mu_1 \times \mu_2$$

where $\nu^i = u_0^i, e_1^i, \ldots, e_{n_i}^i, u_{n_i}^i = \mu^i$ is a path from ν^i to μ^i in P_i, and $n_i \leq \operatorname{diam} P_i$. Further, for vertices ν^i, μ^i in P_i with $\ell(\nu^i, \mu^i) = \operatorname{diam} P_i$, any path connecting $\nu^1 \times \nu^2$ to $\mu^1 \times \mu^2$ can be partitioned into a path connecting ν^1 to μ^1 and a path connecting ν^2 to μ^2, hence the total length of the path is $\operatorname{diam} P_1 + \operatorname{diam} P_2$.

The second inequality can be seen by simply noting that the projection of any path to one of the top or bottom facets produces a path in P, and so any two points ν, μ of distance $\operatorname{diam} P$ apart have $\nu \times \{i\}$, $\mu \times \{j\}$ of distance at least $\operatorname{diam} P$ also $i,j = 0$ or 1.

An outstanding conjecture in polyhedral theory, the Hirsch Conjecture, as discussed in [17] Chapter 16 or [24], claims that for d-polytopes $\operatorname{diam} P \leq n-d$, where n is the number of facets of P. The major polyhedra for which the Hirsch Conjecture is known to hold fall into the following three classes.

1. Transportation polyhedra

$P = \{x \in R_+^{pq} | \sum_{j=1}^{q} x_{ij} = a_i, \; i = 1,\ldots,p, \text{ and } \sum_{i=1}^{p} x_{ij} = b_j,$

$i = 1,\ldots,q$. Balinski [2] proved the Hirsch Conjecture for the cases $a_i = k_i m_1 + 1$ $b_j = m_1$ $k_i \geq 0$ $\sum_{i=1}^{m_1} k_i = m_2 - 1$ $a_i = (k_i + 1)m_1 - 1$ $b_j = m_1$ $k_i \geq 0$ $\sum_{i=1}^{m_1} k_i = m_2 - m_1 + 1$. Balinski and Russakoff [3] proved it for the case $a_i = b_j = 1$, called the Assignment Polytope.

2. Leontief substitution systems

$P = \{x \in \mathbb{R}_+^p | Ax = b\}$, where b is a non-negative m-vector and A is an $m \times p$ pre-Leontief matrix, i.e., A is full row rank and contains at most one positive entry per column. Saigal [32] showed the Hirsch Conjecture for the special case A the node-arc incidence matrix of a directed source-sink network, v the incidence vector of the source and sink, called the Shortest Path Polyhedron. Grinold [16] later showed it for the general case.

3. General polyhedra

Klee and Walkup [24] provide the Hirsch Conjecture for polytopes with $d \leq 3$ and $n-d \leq 5$. They found a class of unbounded polyhedra of all dimensions $d \geq 4$, $n = 2d$ which violated the Hirsch Conjecture. They also showed that it is sufficient to consider only simple polytopes, and only those with $n = 2d$.

Proven upper bounds on the diameter of d-polytopes with n facets are very high. Larman [25] established a bound of $2^{n-3}n$, and later Barnette [6] improved it somewhat. He gives the bound $\frac{1}{3} 2^{d-3} (n-d+\frac{5}{2})$,

which is incorrect for $d = 3$, but his proof seems to yield the bound $\frac{2}{3}\, 2^{d-3}\,(n-d+\frac{3}{2})$.

In a simplicial complex the concept dual to edge paths is that of simplicial paths. Given two d-simplices Δ_0, Δ_1 in a d-dimensional simplicial complex Σ, a <u>simplicial path</u> between Δ_0 and Δ_1 is a sequence of d-simplices $\Delta_0 = \sigma_0, \sigma_1, \ldots, \sigma_k = \Delta_1$ for which $\sigma_i \cap \sigma_{i-1}$ is a (d-1)-face of σ_i and σ_{i-1} $i = 1,\ldots,k$. The <u>length</u> of such a path is k, one less than the number of d-simplices in the path. Then the distance between Δ_0 and Δ_1, $\ell(\Delta_0,\Delta_1)$, is the length of the shortest simplicial path between Δ_0 and Δ_1, and the <u>simplicial diameter</u> of Σ (or just diam Σ) is the maximum of $\ell(\Delta_0,\Delta_1)$ as Δ_0, Δ_1 range over all d-simplices of Σ.

We have immediately from the definition of dual complexes that if P is a simple polyhedron then

$$\text{diam } P = \text{diam } \Sigma_P^*$$

and the corresponding form of the Hirsch Conjecture says that the diameter of the (d-1)-dimensional complex of the boundary of a simplicial d-polytope with n vertices is $n-d$. More generally, we say that any d-dimensional simplicial complex Σ with n vertices satisfies the Hirsch Conjecture if

$$\text{diam } \Sigma \leq n-d-1.$$

CHAPTER 3

k-DECOMPOSABLE COMPLEXES

3.1 Definitions

We introduce the main concept of this thesis here, namely k-decomposability.

Definition 1: A simplicial complex Σ is k-decomposable if Σ is pure dimensional and either $\Sigma = \{\emptyset\}$ or there exists a simplex $\tau \in \Sigma$, with $\dim \tau \le k$ such that

1) $\Sigma \backslash \tau$ is k-decomposable

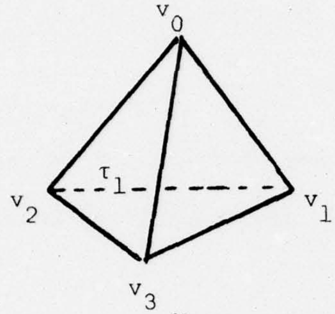
2) $\ell k_{\Sigma} \tau$ is k-decomposable

k-decomposability, then, forms a hierarchy, with k-decomposability implying (k+1)-decomposability for $0 \le k < \dim \Sigma$, and k-decomposability equivalent to (k+1)-decomposability for $k \ge \dim \Sigma$. Since τ of the definition can never be the null simplex ($\Sigma \backslash \emptyset = \emptyset$ is not even a simplicial complex) then k-decomposability is not possible for $k < 0$. The most restrictive case, $k = 0$, is of special importance, and will be called vertex decomposability.

Table 1 illustrates a complete 2-decomposition of the complex $\Sigma = cl\{v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3\}$. One example of a complex which is not k-decomposable for any k is the complex $\Sigma = cl\{v_1v_2, v_3v_4\}$. Deletion of any simplex from Σ results in a complex which is not pure.

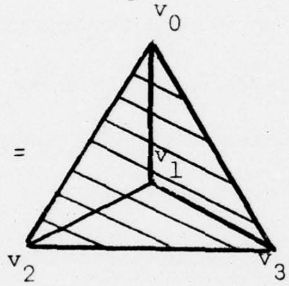
We present now two elementary but important classes of vertex decomposable complexes, namely, the d-simplex and the boundary of a d-simplex.

$\Sigma_0 = \Sigma =$



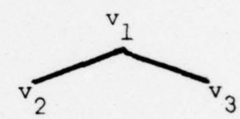
$\tau_1 = v_0v_2v_3, \quad \ell k_{\Sigma_0}\tau_0 = \{\emptyset\}$

$\Sigma_1 = \Sigma_0\backslash\tau_1 =$



$\tau_2 = v_0$

$\hat{\Sigma}_0 = \ell k_{\Sigma_1}\tau_2 =$



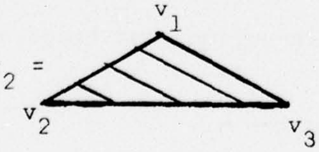
$$\hat{\tau}_1 = v_2 \begin{cases} \hat{\hat{\Sigma}}_0 = \ell k_{\hat{\Sigma}_0}\hat{\sigma}_1 = \overset{v_1}{.} \\ \hat{\hat{\tau}}_1 = v_1 \\ \ell k_{\hat{\hat{\Sigma}}_0}\hat{\hat{\tau}}_1 = \hat{\hat{\Sigma}}_0\backslash\hat{\hat{\tau}}_1 = \{\emptyset\} \end{cases}$$

$\hat{\Sigma}_1 = \hat{\Sigma}_0\backslash\hat{\tau}_1 = v_1 \text{———} v_3$

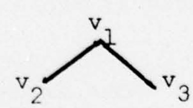
$$\hat{\tau}_2 = v_3 \begin{cases} \hat{\hat{\Sigma}}_0 = \ell k_{\hat{\Sigma}_0}\hat{\tau}_2 = \overset{v_1}{.} \\ \text{etc.} \end{cases}$$

$\hat{\Sigma}_2 = \hat{\Sigma}_1\backslash\hat{\tau}_2 = \overset{v_1}{.}$ etc.

$\Sigma_2 = \Sigma_0\backslash\tau_2 =$



$\tau_3 = v_2v_3$ $\left\{ \ell k_{\Sigma_2}\tau_3 = \right.$



etc.

$\Sigma_3 = \Sigma_2\backslash\tau_3 = \overset{v_1}{.}$ etc.

Table 1

A 2-decomposition of $\Sigma = \mathrm{cl}\{v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3\}$

Proposition 3.1.1: The d-simplex Δ^d $(d \geq -1)$ and its boundary $\partial\Delta^d$ $(d \geq 0)$ are vertex decomposable complexes.

Proof: Recall $\Delta^d = \overline{v_0 \ldots v_d}$ and $\partial\Delta^d = \overline{v_0 \ldots v_d} \setminus v_0 \ldots v_d$. Certainly these are both pure dimensional complexes of dimension d and $d-1$ respectively. We prove both are vertex decomposable by induction on d.

Δ^d: If $d = -1$ then $\Delta^d = \{\emptyset\}$, which is vertex decomposable. Otherwise, choose any v_i in Δ^d. We have $\ell k_{\Delta^d} v_i = \Delta^d \setminus v_i = \overline{v_0 \ldots v_{i-1} v_{i+1} \ldots v_d}$ which is a $(d-1)$-simplex, hence by induction is vertex decomposable. Hence Δ^d is vertex decomposable.

$\partial\Delta^d$: If $d = 0$, then $\partial\Delta^d = \overline{v}_0 \setminus v_0 = \{\emptyset\}$, which is vertex decomposable. Otherwise choose any vertex v_i in Δ^d. We have $\partial\Delta^d v_i = \overline{v_0 \ldots v_{i-1} v_{i+1} \ldots v_d}$ which is a $(d-1)$-simplex, hence by the first part of the proof, is vertex decomposable. And

$$\begin{aligned} \ell k_{\partial\Delta^d} v_i &= \overline{v_0 \ldots v_{i-1} v_{i+1} \ldots v_n} \setminus v_0 \ldots v_{i-1} v_{i+1} \ldots v_n \\ &= \partial\Delta^{d-1} \end{aligned}$$

which is again vertex decomposable by induction, and hence so is $\partial\Delta^d$.

More examples will be given in Chapter 4.

We now give two equivalent definitions which will be of importance throughout the thesis.

Definition 2: A d-dimensional complex Σ is k-decomposable if it is pure dimensional and either Σ is a d-simplex or there exists a simplex

$\tau \in \Sigma$, $\dim \tau \leq k$, such that

1) $\Sigma\backslash\tau$ is d-dimensional and k-decomposable

2) $\ell k_\Sigma \sigma$ is $(d-|\sigma|)$-dimensional and k-decomposable.

Definition 3: A (not necessarily pure) d-dimensional complex Σ is k-decomposable if either Σ is a d-simplex or there exists a simplex $\tau \in \Sigma$, $\dim \tau \leq k$, such that

1) $\Sigma\backslash\tau$ is d-dimensional and k-decomposable

2) $\ell k_\Sigma \tau$ is $(d-|\tau|)$-dimensional and k-decomposable.

Before we prove equivalence of the three definitions, we need to prove a lemma.

Lemma: If Σ is a k-decomposable complex by Definition 2, and σ is a set of elements disjoint from the vertex set $V(\Sigma)$ of Σ, then $\bar{\sigma}.\Sigma$ is k-decomposable by Definition 2.

Proof: We prove the lemma by induction on $|\Sigma|$, the number of simplices in Σ. If $|\Sigma| = 1$ then $\Sigma = \{\emptyset\}$ and $\bar{\sigma}.\Sigma = \bar{\sigma}$ which is a $(|\sigma|-1)$-simplex, hence k-decomposable. So let $|\Sigma| > 1$ and $\dim \Sigma = d$. Certainly $\bar{\sigma}.\Sigma$ is pure $(d+|\sigma|)$-dimensional, since Σ is pure d-dimensional. If Σ is a d-simplex, then $\bar{\sigma}.\Sigma$ is a $(d+|\sigma|)$-simplex, hence again k-decomposable. Otherwise, there must exist a $\tau \in \Sigma$, $\dim \tau \leq k$, such that $\Sigma\backslash\tau$ is d-dimensional and k-decomposable by Definition 2, and $\ell k_\Sigma \tau$ is $(d-|\tau|)$-dimensional and k-decomposable by Definition 2. Further, $|\Sigma\backslash\tau|$ and $|\ell k_\Sigma \tau|$ are both less than $|\Sigma|$, so by induction $\bar{\sigma}.(\Sigma\backslash\tau)$ and $\bar{\sigma}.\ell k_\Sigma \tau$ are both k-decomposable by Definition 2. But by Lemma 2.3.2, $\bar{\sigma}.(\Sigma\backslash\tau) = (\bar{\sigma}.\Sigma)\backslash\tau$ and

$\bar{\sigma}.\ell k_{\Sigma}\tau = \ell k_{\bar{\sigma}.\Sigma}\tau$. Therefore τ satisfies the requirements for Definition 2 on the complex $\bar{\sigma}.\Sigma$, and hence $\bar{\sigma}.\Sigma$ is k-decomposable by Definition 2. This completes the proof of the lemma.

Proof of equivalence of Definitions 1, 2, and 3: Clearly Definition 2 implies Definition 1 (since d-simplices are k-decomposable), and Definition 2 implies Definition 3. We prove Definition 1 implies Definition 2, and Definition 3 implies Definition 2.

$(1 \Rightarrow 2)$: We proceed by induction on $|\Sigma|$. If $|\Sigma| = 1$ then $\Sigma = \{\emptyset\}$ which is a (-1)-simplex and hence k-decomposable by Definition 2. Otherwise let Σ be d-dimensional and k-decomposable by Definition 1, $|\Sigma| \geq 2$, so that there exists a $\tau \in \Sigma$, $\dim \sigma \leq k$ such that $\ell k_{\Sigma}\tau$ and $\Sigma \backslash \tau$ are k-decomposable. Σ pure d-dimensional implies that $\ell k_{\Sigma}\tau$ is $(d-|\tau|)$-dimensional automatically. If $\Sigma \backslash \tau$ is d-dimensional then we are done. Otherwise it must be that τ is contained in every d-simplex of Σ, so that $\Sigma = \bar{\tau}.\ell k_{\Sigma}\tau$. But $\ell k_{\Sigma}\tau$ is k-decomposable by Definition 1, hence by induction is k-decomposable by Definition 2. Therefore from the lemma we see that Σ is also k-decomposable by Definition 2.

$(2 \Rightarrow 3)$: All that needs to be proved here is that if Σ is d-decomposable by Definition 3 then it is pure dimensional. We proceed by induction on $|\Sigma|$. If $|\Sigma| = 1$ then $\Sigma = \{\emptyset\}$, which is certainly pure. Otherwise, let Σ be d-dimensional and k-decomposable by Definition 3, $|\Sigma| > 2$. If Σ is a d-simplex then Σ is certainly pure dimensional. Otherwise there must exist a simplex $\tau \in \Sigma$ with

$\Sigma \setminus \tau$ and $\ell k_\Sigma \tau$ both k-decomposable by Definition 3. But then by induction on $|\Sigma \setminus \tau| < |\Sigma|$ and $|\ell k_\Sigma \tau| < |\Sigma|$, $\Sigma \setminus \tau$ and $\ell k_\Sigma \tau$ are both pure of dimension d and $d-|\tau|$ respectively. Hence $\Sigma = (\Sigma \setminus \tau) \cup \bar{\tau}.\ell k_\Sigma \tau$ is the union of two pure d-dimensional complexes, and is therefore itself pure.

The τ referred to in the definitions will be called a <u>shedding simplex</u>. Although the property of being k-decomposable is independent of which definition is used, the properties of the shedding simplices and of the resulting complexes may vary from one definition to the next (in particular between Definition 1 and Definitions 2 or 3). We will always specify which definition we will be working with when such a distinction is needed. A <u>shedding order</u> is a sequence of simplices $\tau_1,\ldots,\tau_m$, $|\tau| \leq k$, *defined inductively so that*

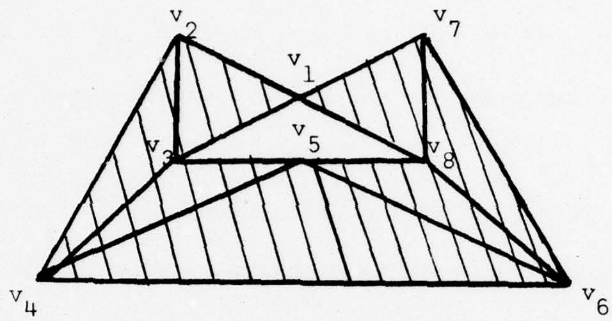
1) τ_1 is a shedding simplex for $\Sigma_0 \equiv \Sigma$
2) τ_{i+1} is a shedding simplex for $\Sigma_i \equiv \Sigma_{i-1} \setminus \tau_i$, $i = 2,\ldots,m-1$
3) Σ_m is a d-simplex (or $\{\emptyset\}$).

In other words, Σ_i $i = 1,\ldots,m$ are pure (d-dimensional) and $\ell k_{\Sigma_i} \tau_i$ is k-decomposable. Again, refer to Table 1 for illustration.

To close the section, we define a slightly wider class of complexes, the <u>weakly k-decomposable</u> complexes, by dropping the condition on the links.

<u>Definition 1^W</u>: A simplicial complex Σ is <u>weakly k-decomposable</u> if Σ is pure and either $\Sigma = \{\emptyset\}$ or there is a simplex $\tau \in \Sigma$, $\dim \tau \leq k$ such that $\Sigma \setminus \tau$ is weakly k-decomposable. Weak vertex decompositions, shedding simplices, and shedding orders are defined analogously.

Certainly k-decomposable complexes are weakly k-decomposable. The converse is not true. For example, the complex



is weakly vertex decomposable with shedding order $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$, but this is not the shedding order of a vertex decomposition (since $\ell k_\Sigma v_1 = cl\{v_2v_3, v_7v_8\}$ is not vertex decomposable). In fact, this complex is not vertex decomposable at all, since removal of any vertex except v_1 results in a complex which is not pure. The complex $\Sigma = cl\{v_1v_2, v_3v_4\}$, mentioned earlier, is also an example of a complex which is not weakly k-decomposable for any k.

There is likewise a weak version of Definition 2.

Definition 2^W: A simplicial complex Σ is weakly k-decomposable if Σ is pure d-dimensional and either Σ is a d-simplex or there is a simplex $\tau \in \Sigma$, $\dim \tau \leq k$, such that $\Sigma \backslash \tau$ is a d-dimensional weakly k-decomposable complex.

Definition 1^W is equivalent to Definition 2^W, the proof being contained in that equating Definition 1 and Definition 2. In fact, many of the proofs of properties held by weakly and strongly k-decomposable complexes will proceed simultaneously. Definition 3, of course, has no equivalent weak analogue.

3.2 Some Properties of k-decomposability

We prove in this section that many of the operations defined in Chapter 1 preserve k-decomposition.

Proposition 3.2.1: The link of every simplex of a k-decomposable complex is itself k-decomposable.

Proof: We proceed by induction on $|\Sigma|$. If $|\Sigma| = 1$ then $\Sigma = \{\emptyset\}$, of which the single link $\ell k_\Sigma \emptyset = \{\emptyset\}$ is k-decomposable. Otherwise, let τ be a shedding simplex for Σ, so that $\Sigma\backslash\tau$ and $\ell k_\Sigma \tau$ are both k-decomposable, and choose σ a simplex in Σ. We prove $\ell k_\Sigma \tau$ is k-decomposable.

Case 1 ($\tau \cup \sigma \notin \Sigma$): We have $\ell k_\Sigma \sigma = \ell k_{\Sigma\backslash\tau} \sigma$, which is k-decomposable by induction on $|\Sigma\backslash\tau| < |\Sigma|$.

Case 2 ($\tau \subseteq \sigma$): By Lemma 2.3.4 we have

$$\ell k_\Sigma \sigma = \ell k_{\ell k_\Sigma \tau}(\sigma\backslash\tau),$$

which is k-decomposable by induction on $|\ell k_\Sigma \tau| < |\Sigma|$.

Case 3 ($\sigma \cup \tau \in \Sigma$, $\tau \not\subseteq \sigma$): We prove that $\tau\backslash\sigma$ is a shedding simplex for $\ell k_\Sigma \sigma$. By Lemma 2.3.3 and Lemma 2.3.5 we have

$$(\ell k_\Sigma \sigma)\backslash(\tau\backslash\sigma) = \ell k_{\Sigma\backslash\tau} \sigma,$$

which is k-decomposable as in Case 1. Further, by Lemma 2.3.4 we have

$$\ell k_{\ell k_\Sigma \sigma}(\tau\backslash\sigma) = \ell k_\Sigma(\tau \cup \sigma) = \ell k_{\ell k_\Sigma \tau}(\sigma\backslash\tau),$$

which is k-decomposable as in Case 2. Hence $\tau\backslash\sigma$ is indeed a shedding simplex for $\ell k_\Sigma \sigma$. This completes the proof.

Proposition 3.2.2: (Weak) k-decomposability is preserved under joins.

Proof: Let Σ_1, Σ_2 be d_1- and d_2-dimensional (weakly) k-decomposable complexes. Then Σ_1 and Σ_2 are pure dimensional, hence so is $\Sigma_1 . \Sigma_2$. Proceed by induction on $|\Sigma_1 . \Sigma_2|$. If $|\Sigma_1 . \Sigma_2| = 1$ then $\Sigma_1 . \Sigma_2 = \{\emptyset\}$, which is k-decomposable. Otherwise one of the complexes, say Σ_1, has $\Sigma_1 \neq \emptyset$. Hence there must be a simplex $\tau \in \Sigma_1$ so that $\Sigma_1 \backslash \tau$ and $\ell k_\Sigma \tau$ are both (weakly) k-decomposable. But by Lemma 2.3.2 $(\Sigma_1 . \Sigma_2)\backslash\tau = (\Sigma_1\backslash\tau) . \Sigma_2$ and $\ell k_{\Sigma_1 . \Sigma_2} \tau = (\ell k_{\Sigma_1} \tau) . \Sigma_2$, and so by induction on $|(\Sigma_1\backslash\tau) . \Sigma_2| < |\Sigma_1 . \Sigma_2|$ and $|(\ell k_{\Sigma_1} \tau) . \Sigma_2| < |\Sigma_1 . \Sigma_2|$, $\Sigma_1 . (\Sigma_2\backslash\tau)$ and $\ell k_{\Sigma_1 . \Sigma_2} \tau$ are both (weakly) k-decomposable. Hence τ is also a shedding simplex for $\Sigma_1 . \Sigma_2$ (by Definition 1) and so $\Sigma_1 . \Sigma_2$ is k-decomposable.

Proposition 3.2.3: (Weak) k-decomposability is preserved under stellar subdivision.

The proof to this proposition is too tedious to include here; it can be found in Appendix 1. We do give a more enlightening proof for the special case of vertex decomposition.

Proposition 3.2.4: (Weak) vertex decomposition is preserved under stellar subdivision.

Proof: Let Σ be (weakly) vertex decomposable under Definition 1, $X \neq \emptyset$ a simplex in Σ, and a the additional vertex. Recall the

definition

$$\begin{aligned} st(a,X) &= (\Sigma\backslash X) \cup (a.\partial X.\ell k_{\Sigma} X) \\ &= (\Sigma\backslash X) \cup (\bar{a}.\partial X.\ell k_{\Sigma} X). \end{aligned}$$

We have $|\Sigma| \geq 2$ and if $\Sigma = \bar{\sigma}$ is a simplex, then

$$\begin{aligned} st(a,X)[\Sigma] &= (\bar{\sigma}\backslash X) \cup (a.\partial X.\ell k_{\bar{\sigma}} X) \\ &= (\overline{\sigma\backslash X}) \cup a.\partial X.(\overline{\sigma\backslash X}) \\ &= \bar{a}.\partial X.(\overline{\sigma\backslash X}) \end{aligned}$$

which is (weakly) vertex decomposable, since each component is (Proposition 3.2.2). Further, if $X = \{u\}$ is a vertex in Σ, then

$$\begin{aligned} st(a,X)[\Sigma] &= (\Sigma\backslash u) \cup (a.\{\emptyset\}.\ell k_{\Sigma} u) \\ &= \Sigma\backslash u \cup a.\ell k_{\Sigma} u \end{aligned}$$

which is merely Σ with u relabeled as a, and hence vertex decomposable.

Therefore, proceed by induction on $|\Sigma| > 2$ with the assumption that Σ is not a simplex nor X a vertex, and let v be a shedding simplex for Σ. We take three cases:

Case 1 ($v \not\in X$, $v \not\in \ell k_{\Sigma} X$): We have

$$st(a,X)[\Sigma]\backslash v = [(\Sigma\backslash X) \cup \bar{a}.\partial X.\ell k_\Sigma X]\backslash v$$

$$= [(\Sigma\backslash X)\backslash v] \cup [\bar{a}.\partial X.\ell k_\Sigma X] \quad \text{(Lemma 2.3.1i)}$$

$$= [(\Sigma\backslash v)\backslash x] \cup \bar{a}.\partial X.\ell k_{\Sigma\backslash v} X \quad \text{by Lemma 2.3.3i and the fact that } v \notin \ell k_\Sigma X.$$

$$= st(a,X)[\Sigma\backslash v]$$

which is (weakly) vertex-decomposable by induction on $|\Sigma\backslash v| < |\Sigma|$. (Further

$$\ell k_{st(a,X)[\Sigma]} v = \ell k_{\Sigma\backslash X} v \quad \text{(Lemma 2.3.1)}$$

$$= \ell k_\Sigma v \quad (\text{since } v.X \not\in \Sigma)$$

which is (weakly) k-decomposable by choice of v.)

Case 2 ($v \in \ell k_\Sigma X$): We have

$$st(a,X)[\Sigma]\backslash v = [(\Sigma\backslash X)\backslash v] \cup [(\bar{a}.\partial X.\ell k_\Sigma X)\backslash v] \quad \text{(Lemma 2.3.1i)}$$

$$= (\Sigma\backslash v)\backslash X \cup \bar{a}.\partial X.(\ell k_{\Sigma\backslash v} X) \quad \text{(Lemma 2.3.2 and Lemma 2.3.3)}$$

$$= st(a,X)[\Sigma\backslash v]$$

which is vertex decomposable as in Case 1. (Further

$$\ell k_{st(a,X)[\Sigma]}v = \ell k_{(\Sigma\backslash X)}v \cup \ell k_{\bar{a}.\partial X.\ell k_\Sigma X}v \quad \text{(Lemma 2.3.1)}$$

$$= [(\ell k_\Sigma v)\backslash X] \cup [\bar{a}.\partial X.\ell k_{\ell k_\Sigma X}v] \quad \text{(Lemma 2.3.2 and Lemma 2.3.3)}$$

$$= [(\ell k_\Sigma v)\backslash X] \cup [\bar{a}.\partial X.\ell k_{\ell k_\Sigma v}X] \quad \text{(by two applications of Lemma 2.3.4)}$$

$$= st(a,X)[\ell k_\Sigma v],$$

which is vertex decomposable by induction on $|\ell k_\Sigma v| < |\Sigma|$.)

Case 3 $(v \in X \neq \{v\})$: We will shed i) v and then ii) a.

i) $st(a,X)[\Sigma]\ v = (\Sigma\backslash X) \cup \bar{a}.(\partial X\backslash v).\ell k_\Sigma X$ (by Lemma 2.3.1 and 2.3.2) which is pure dimensional, since for Δ a d-simplex in (pure d-dimensional) Σ containing v, either $X \not\subseteq \Delta$, implying $\Delta \in \Sigma\backslash X \subseteq st(a,X)[\Sigma]\backslash v$ or $X \subseteq \Delta$, implying $a.(\Delta\backslash v) \in \bar{a}.(\partial X\backslash v).\ell k_\Sigma X \subseteq st(a,X)[\Sigma]$. (Further

$$\ell k_{st(a,X)[\Sigma]}v = \ell k_{\Sigma\backslash X}v \cup \ell k_{\bar{a}.\partial X.\ell k_\Sigma X}v \quad \text{(Lemma 2.3.1)}$$

$$= [(\ell k_{\Sigma\backslash(X\backslash v)}v] \cup \bar{a}.\ell k_{\partial X}v.\ell k_\Sigma X \quad \text{(Lemma 2.3.2, Lemma 2.3.5)}$$

$$= [(\ell k_\Sigma v)\backslash(X\backslash v)] \cup \bar{a}.\partial(X\backslash v).\ell k_{\ell k_\Sigma v}X \quad \text{(Lemma 2.3.3, Lemma 2.3.4)}$$

$$= st(a,X\backslash v)[\ell k_\Sigma v]$$

which is vertex decomposable by induction on $|\ell k_\Sigma v| < |\Sigma|$.)

ii) $(st(a,X)[\Sigma]\backslash v)\backslash a = [\Sigma\backslash X] \cup [(\bar{a}.(\partial X\backslash v).\ell k_{\Sigma} X)\backslash a]$ (Lemma 2.3.1)

$$= \Sigma\backslash X \cup (\partial X\backslash v).\ell k_{\Sigma} X$$

$$= \Sigma\backslash X \cup (\overline{X\backslash v}).\ell k_{\Sigma} X$$

$$= \Sigma\backslash v,$$

which is vertex decomposable by choice of v. (Further

$$\ell k_{st(a,X)[\Sigma]\backslash v}\, a = \ell k_{\bar{a}.\partial X\backslash v.\ell k_{\Sigma} v}\, a \quad \text{(Lemma 2.3.1)}$$

$$= (\ell k_{\bar{a}} a).(\partial X\backslash v).\ell k_{\Sigma} X \quad \text{(Lemma 2.3.2)}$$

$$= (\overline{X\backslash v}).\ell k_{\Sigma} X$$

which is vertex decomposable by Proposition 3.2.2, since $\overline{X\backslash v}$ is a simplex, hence vertex decomposable and $\ell k_{\Sigma} X$ is shed by Proposition 3.2.1.) This completes the proof of Proposition 3.2.4.

Proposition 3.2.5: (Weak) vertex decomposability are preserved under wedging.

Proof: Let Σ be a (weakly) vertex decomposable complex by Definition 1, v a vertex in Σ. Then $|\Sigma| \geq 2$ and if $|\Sigma| = 2$ then $\Sigma = \bar{v}$ and $w(\Sigma,v) = \overline{ab}$ which is vertex decomposable. Otherwise proceed by induction on $|\Sigma| > 2$. Let u be a shedding vertex of Σ. Again, recall $w(\Sigma,v) = \{a,b,\emptyset\}.(\Sigma\backslash v) \cup ab.\ell k_{\Sigma} v = \{a,b,\emptyset\}.(\Sigma\backslash v)$ $\overline{ab}.\ell k_{\Sigma} v$. Let u be the shedding vertex of Σ. Again, we take three cases.

Case 1 ($u \neq v$, $u \notin \ell k_\Sigma v$): We have

$$w(\Sigma,v)\backslash u = [\{a,b,\emptyset\}.(\Sigma\backslash v)\backslash u] \cup \overline{ab}.\ell k_\Sigma v \quad \text{(Lemma 3.2.1)}$$

$$= \{a,b,\emptyset\}.(\Sigma\backslash v\backslash u) \cup \overline{ab}.\ell k_\Sigma v \quad \text{(Lemma 2.3.2)}$$

$$= \{a,b,\emptyset\}.(\Sigma\backslash u\backslash v) \cup \overline{ab}.\ell k_{\Sigma\backslash u} v \quad \text{(Since } uv \notin \Sigma)$$

$$= w(\Sigma\backslash u,v)$$

which is (weakly) vertex decomposable by induction on $|\Sigma\backslash u| < |\Sigma|$. (Further,

$$\ell k_{w(\Sigma,v)} u = \{a,b,\emptyset\}.\ell k_{\Sigma\backslash v} u \quad \text{(Lemma 2.3.1, Lemma 2.3.2)}$$

$$= \{a,b,\emptyset\}.\ell k_\Sigma u \quad \text{(since } uv \notin \Sigma)$$

which is vertex decomposable since both factors are.) So η is also a shedding simplex for $w(\Sigma,v)$.

Case 2 ($u \in \ell k_\Sigma v$): We have

$$w(\Sigma,v)\backslash u = \{a,b,\emptyset\}.(\Sigma\backslash v\backslash u) \cup \overline{ab}.(\ell k_\Sigma v\backslash u) \quad \text{(Lemma 2.3.1, 2.3.2)}$$

$$= \{a,b,\emptyset\}.(\Sigma\backslash u\backslash v) \cup \overline{ab}.\ell k_{\Sigma\backslash u} v \quad \text{(Lemma 2.3.3)}$$

$$= w(\Sigma\backslash u,v)$$

which is (weakly) vertex decomposable as in Case 1. (Further,

$$\ell k_{w(\Sigma,v)}u = \{a,b,\emptyset\}.\ell k_{\Sigma\backslash v}u \cup \overline{ab}.\ell k_{\ell k_\Sigma v}u \quad \text{(Lemmas 2.3.1, 2.3.2)}$$

$$= \{a,b,\emptyset\}.[(\ell k_\Sigma u)\backslash v] \cup \overline{ab}.\ell k_{\ell k_\Sigma u}v$$

by Lemma 2.3.3 and two applications of Lemma 2.3.4

$$= w(\ell k_\Sigma u,v)$$

which is vertex decomposable by induction on $|\ell k_\Sigma u| < |\Sigma|$.) So u is again a shedding simplex for $w(\Sigma,v)$.

Case 3 $(u = v)$: Then

$$w(\Sigma,v)\backslash a = (\{a,b,\emptyset\}\backslash a).(\Sigma\backslash v) \cup (\overline{ab}\backslash a).\ell k_\Sigma v \quad \text{(Lemma 2.3.1, Lemma 2.3.2)}$$

$$= \overline{b}.(\Sigma\backslash v) \cup \overline{b}.\ell k_\Sigma v$$

$$= \overline{b}.(\Sigma\backslash v)$$

which is (weakly) vertex decomposable by choice of $u = v$. (Further,

$$\ell k_{w(\Sigma,v)}a = (\ell k_{\{a,b,\emptyset\}}a).(\Sigma\backslash v) \cup (\ell k_{\overline{ab}}\, a).\ell k_\Sigma v \quad \text{(Lemma 2.3.1, Lemma 2.3.2)}$$

$$= (\Sigma\backslash v) \cup \overline{b}.\ell k_\Sigma v$$

$$= \Sigma \quad \text{(by replacing } v \text{ with } b\text{)}$$

which is vertex decomposable.) Hence a is a shedding simplex for $w(\Sigma,v)$.

This exhausts the possible choices of v, and thus proves the theorem.

3.3 k-Decomposability and Shellability

This section will be denoted to proving the following Theorem:

Theorem 3.3: A d-dimensional complex Σ is d-decomposable iff Σ is shellable. An immediate corollary is:

Corollary: Every k-decomposable complex is shellable. Note that weak k-decomposability cannot insure shellability, as the example at the end of section 3.1 shows.

Proof of Theorem: ($\Leftarrow$) Let Σ be a d-dimensional complex, $\sigma_1,\dots,\sigma_m$ an ordering of the d-simplices of Σ which shells Σ, so that $\Sigma = \mathrm{cl}(\bigcup_{i=1}^{m} \sigma_i)$. We prove that Σ is strongly d-decomposable by Definition 2. If $m = 1$ then Σ is a d-simplex, and so Σ is k-decomposable. For $m > 1$ we proceed by induction on $|\Sigma|$. We have by definition that $\bar{\sigma}_m \cap (\bigcup_{i=1}^{m-1} \bar{\sigma}_i)$ is a pure (d-1)-dimensional complex generated by the (d-1)-simplices $\tau_1,\dots,\tau_k$. Let $\tau = \bigcup_{j=1}^{k} (\sigma_m/\tau_j)$. Then $\tau \subseteq \sigma_m$ (hence $\tau \in \Sigma$), but $\tau \not\subseteq \sigma_i$ $i < m$, since it cannot be that $\tau = \tau \cap \sigma_i \subseteq \sigma_m \cap \sigma_i \subseteq \tau_j$, for some j, while $\emptyset \neq \sigma_m \backslash \tau_j \subseteq \tau$. Therefore σ_m is the only d-simplex containing τ. So

1)
$$\Sigma\backslash\tau = \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup (\bar{\sigma}_m\backslash\tau)$$

$$= \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup (\overline{\sigma_m\backslash\tau})$$

$$= \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup \mathrm{cl}\left(\sigma_m\backslash \bigcup_{j=1}^{k} (\sigma_m\backslash\tau_j)\right)$$

$$= \left(\bigcup_{i=1}^{m-1} \bar{\sigma}_i\right) \cup \mathrm{cl}\left(\bigcup_{j=1}^{k} \tau_j\right)$$

$$= \bigcup_{i=1}^{m-1} \bar{\sigma}_i$$

which is a shellable complex, with shelling order $\sigma_1,\ldots,\sigma_{m-1}$; hence d-decomposable by induction on $|\Sigma\backslash\tau| < |\Sigma|$; and

2)
$$\ell k_\Sigma \tau = \ell k_{\bar{\sigma}_m} \tau$$

$$= (\overline{\sigma_m\backslash\tau})$$

which is a simplex, hence d-decomposable. Therefore τ is a shedding simplex for Σ, and hence Σ is strongly d-decomposable.

($\Rightarrow$) By induction on $|\Sigma|$. If $|\Sigma| = 1$ then $\Sigma = \{\emptyset\}$, which is shellable. Otherwise suppose Σ is d-dimensional and d-decomposable by Definition 2. If Σ is a d-simplex, then Σ is shellable. Otherwise there must exist a simplex $\tau \in \Sigma$ such that 1) $\Sigma\backslash\tau$ is d-dimensional and d-decomposable and 2) $\ell k_\Sigma\tau$ is $(d-|\tau|)$-dimensional and d-decomposable, implying $\ell k_\Sigma\tau$ is $(d-|\tau|)$-decomposable.

1) implies $\Sigma\backslash\tau$ is shellable, by induction on $|\Sigma\backslash\tau| < |\Sigma|$, with shelling order $\sigma_1,\ldots,\sigma_p$ of the d-simplexes in Σ not containing τ.
2) implies $\ell k_\Sigma \tau$ is shellable, by induction on $|\ell k_\Sigma \tau| < |\Sigma|$, with shelling order $\tau_1,\ldots,\tau_\ell$. Let $\sigma_{p+i} = \tau \cup \tau_i$, $i = 1,\ldots,\ell$. We claim that $\sigma_1,\ldots,\sigma_{p+\ell}$ is a shelling for Σ. For $2 \leq k \leq p$ $\bar{\sigma}_k \cap (\bigcup_{i=1}^{k-1} \bar{\sigma}_i)$ is a pure (d-1)-dimensional complex by(1). For $k > p$, we have

$$\begin{aligned}
\bar{\sigma}_k \cap \left(\bigcup_{i=1}^{k-1} \bar{\sigma}_i\right) &= \left[\bar{\sigma}_k \cap \left(\bigcup_{i=1}^{p} \bar{\sigma}_i\right)\right] \cup \left[\bar{\sigma}_k \cap \left(\bigcup_{i=p+1}^{k-1} \bar{\sigma}_i\right)\right] \\
&= [\bar{\sigma}_k \cap (\Sigma\backslash\tau)] \cup \left[(\bar{\tau}.\bar{\tau}_{k-p}) \cap \left(\bigcup_{i=1}^{k-p} (\bar{\tau}.\bar{\tau}_i)\right)\right] \\
&= [\bar{\sigma}_k\backslash\tau] \cup \left[\bar{\tau}.\left(\bar{\tau}_{k-p} \cap \left(\bigcup_{i=1}^{k-p} \bar{\tau}_i\right)\right)\right].
\end{aligned}$$

Now $\bar{\sigma}_k\backslash\tau = \bigcup_{v\in\tau} \overline{\sigma\backslash v}$, which is pure of dimension d-1, and $\bar{\tau}_{k-p} \cap (\bigcup_{i=1}^{k-p} \bar{\tau}_i)$ is pure of dimension $d-|\tau|-1$, since $\tau_1,\ldots,\tau_\ell$ is a shelling order. Hence $\bar{\sigma}_k \cap (\bigcup_{i=1}^{k-1} \bar{\sigma}_i)$ is a pure (d-1)dimensional complex for $k = 2,\ldots,p+\ell$, and therefore $\Sigma = \bigcup_{i=1}^{p+\ell} \bar{\sigma}_i$ is shellable.

We observe the important special case that the simplicial duals of convex polyhedra and polytopes fall at least into the bottom of the hierarchy of k-decomposable complexes. It remains to be seen how far up in the hierarchy they lie.

3.4 k-Decomposability and Diameters

It turns out that k-decomposability and weak d-decomposability force bounds on the (simplicial) diameter of simplicial complexes.

Theorem 3.4.1: If Σ is a d-dimensional k-decomposable complex, $0 \le k \le d$, then

$$\operatorname{diam} \Sigma \le f_k(\Sigma) - \binom{d+1}{k+1}$$

where $f_k(\Sigma)$ is the number of k-faces of Σ.

Proof: We proceed by induction on $|\Sigma| \ge 2$ (since $\dim \Sigma \ge 0$). If $|\Sigma| = 2$ then $\Sigma = \bar{v}$, and so $\dim \Sigma = 0$, $k = 0$, $f_0(\Sigma) = 1$, and $\operatorname{diam} \Sigma = 0 \le f_0(\Sigma) - \binom{d+1}{k+1}$. Otherwise proceed by induction on $|\Sigma| > 2$.

Let Δ_0, Δ_1 be two d-simplices in Σ, and τ a shedding simplex for Σ.

Case 1 ($\tau \nsubseteq \Delta_1 \cap \Delta_2$): Then at least one of Δ_1, Δ_2 is in $\Sigma \backslash \tau$, say Δ_2. This implies $\Sigma \backslash \tau$ is pure d-dimensional, and so for $v \in \tau$, $\Delta_1 \backslash v \in \Sigma \backslash \tau$ must be contained in some d-simplex $\Delta_1' \in \Sigma \backslash \tau$, so that Δ_1, Δ_1' are adjacent. But $\Sigma \backslash \tau$ is k-decomposable, so by induction on $|\Sigma \backslash \tau| < |\Sigma|$

$$\begin{aligned} \operatorname{diam}(\Sigma \backslash \tau) &\le f_k(\Sigma \backslash \tau) - \binom{d+1}{k+1} \\ &\le f_k(\Sigma) - 1 - \binom{d+1}{k+1} \end{aligned}$$

(since τ must be in at least one k-simplex) and so Δ_1' can be joined to Δ_2 by a simplicial path of length at most $f_k(\Sigma) - \binom{d+1}{k+1} - 1$. Hence Δ_1 can be joined to Δ_2 by a path of length at most $f_k(\Sigma) - \binom{d+1}{k+1}$.

Case 2 $(\tau \subseteq \Delta_1 \cap \Delta_2)$: We have $\Delta_1 \setminus \tau, \Delta_2 \setminus \tau$ in $\ell k_\Sigma \tau$, a k-decomposable complex with $|\ell k_\Sigma \tau| < |\Sigma|$. So, by induction,

$$\operatorname{diam}(\ell k_\Sigma \tau) \leq f_k(\ell k_\Sigma \tau) - \binom{d-|\tau|+1}{k+1}.$$

Now τ is contained in some d-simplex $\Delta \in \Sigma$ which contains $\binom{d+1}{k+1}$ k-faces, $\binom{k+1-|\tau|}{k+1}$ of which do not intersect τ. Hence

$$f_k(\ell k_\Sigma \tau) \leq f_k(\Sigma) - \left[\binom{d+1}{k+1} - \binom{d+1-|\tau|}{k+1}\right]$$

and so

$$\operatorname{diam}(\ell k_\Sigma \tau) \leq f_k(\ell k_\Sigma \tau) - \binom{d-|\tau|+1}{k+1} \leq f_k(\Sigma) - \binom{d+1}{k+1}.$$

Hence $\Delta_1 \setminus \tau, \Delta_2 \setminus \tau$ can be joined by a simplicial path

$$\Delta_1 \setminus \tau = \sigma_1, \ldots, \sigma_n = \Delta_2 \setminus \tau$$

in $\ell k_\Sigma \tau$ of length at most $f_k(\Sigma) - \binom{d+1}{k+1}$, and therefore Δ_1 and Δ_2 can be joined by the path

$$\Delta_1 = \sigma_1 \cup \tau, \sigma_2 \cup \tau, \ldots, \sigma_n \cup \tau = \Delta_2$$

of the same length.

Since these cover all choices of Δ_1, Δ_2 we have $\operatorname{diam} \Sigma \leq f_k(\Sigma) - \binom{d+1}{k+1}$.

Corollary 3.4.2: Vertex decomposable complexes satisfy the Hirsch Conjecture.

Theorem 3.4.3: If Σ is a weakly k-decomposable complex, $0 \leq k \leq d$, then $\operatorname{diam} \Sigma \leq 2f_k(\Sigma)$.

Proof: Let Σ be d-dimensional, weakly k-decomposable by Def 2^w. If Σ is a d-simplex, then $\operatorname{diam} \Sigma = 0 < 2f_k(\Sigma)$. Otherwise, proceed by induction on $|\Sigma| > 2^{(d+1)}$. Let Δ_1, Δ_2 be d-simpleces in Σ, and let τ be a shedding simplex for Σ.

If $\tau \in \Delta_i$, $i = 1$ or 2, then for $v \in \tau$ $\Delta_i \backslash v \in \Sigma\backslash\tau$ must be contained in some d-simplex in $\Sigma\backslash\tau$ (since $\Sigma\backslash\tau$ is pure d-dimensional). Hence there are simplices Δ_1', Δ_2' in $\Sigma\backslash\tau$ which share a (d-1)-face with Δ_1, Δ_2, respectively. But $|\Sigma\backslash\tau| < |\Sigma|$, so by induction

$$\begin{aligned} \operatorname{diam}(\Sigma\backslash\tau) &\leq 2f_k(\Sigma\backslash\tau) \\ &\leq 2[f_k(\Sigma) - 1] \end{aligned}$$

since τ must be contained in at least one k-simplex. Therefore Δ_1', Δ_2' can be joined by a path of length at most $2f_k(\Sigma) - 2$, and so Δ_1, Δ_2 can be joined by a path of length at most $2f_k(\Sigma)$. Hence

$$\operatorname{diam} \Sigma \leq 2f_k(\Sigma).$$

Corollary 3.4.4: (Weakly) k-decomposable complexes have diameters bounded above by a polynomial in n of degree $k+1$.

Proof: $f_k(\Sigma)$ is at most the number of $(k+1)$-sets in the n-set $V(\Sigma)$. Hence $f_k(\Sigma) \leq \binom{n}{k+1}$, a polynomial in n of degree $k+1$. The corollary follows.

CHAPTER 4

CLASSES OF VERTEX DECOMPOSABLE COMPLEXES

4.1 Complexes of Dimension ≤ 3

Proposition 4.1.1: All 0-dimensional complexes are vertex decomposable.

Proof: Simply note that 0-dimensional complexes are point sets, and so for each $v \in \Sigma$, $\ell k_{\Sigma} v = \{\emptyset\}$ and $\Sigma \backslash v$ is either $\{\emptyset\}$ or again a point set. Hence the vertices of Σ can be shed in any order.

Theorem 4.1.2: Let Σ be a 1-dimensional complex, i.e., a loopless graph with at least one edge and no multiple edges. Then the following are equivalent.

1) Σ is connected

2) Σ is vertex decomposable

3) Σ is weakly vertex decomposable

4) Σ is 2-decomposable

5) Σ is weakly 2-decomposable.

Proof: First note that the link of a non-empty simplex in Σ is either $\{\emptyset\}$ or a 0-dimensional complex, both of which are k-decomposable for $k = 1,2$. Hence we have 2 equivalent to 3, 4 equivalent to 5, and 3 implies 4. We prove 5 implies 1 and 1 implies 2.

($5 \Rightarrow 1$): Proceed by induction on $|\Sigma| \geq 4$. If $|\Sigma| = 4$ then Σ is a single edge, which is connected. Otherwise let $|\Sigma| > 4$, so that Σ has a shedding simplex τ. Suppose that Σ contains two non-empty components Σ_1 and Σ_2.

Case 1 ($\tau \notin \Sigma_1 \cup \Sigma_2$): Here $\Sigma\backslash\tau$ still has components Σ_1 and Σ_2, and $|\Sigma\backslash\tau| < |\Sigma|$. But $\Sigma\backslash\tau$ is weakly 2-decomposable, and so by induction is connected. This contradicts the existence of Σ_1 and Σ_2.

Case 2 ($\tau \in \Sigma_1$): Σ pure 2-dimensional implies that τ must be contained in an edge e of Σ_1, and so e must contain a vertex v which is not τ. But then $v \in \Sigma_1\backslash\tau$, and hence $\Sigma\backslash\tau$ still has non-empty components $\Sigma_1\backslash\tau$, Σ_2. But again $\Sigma\backslash\tau$ is weakly decomposable, contradicting the existence of Σ_1 and Σ_2.

The case $v \in \Sigma_2$ is handled similarly.

($1 \Rightarrow 2$): We have automatically that Σ connected implies Σ pure dimensional, since there can be no isolated points in Σ. We proceed again by induction on $|\Sigma| \geq 4$ if $|\Sigma| = 4$ then Σ is a single edge, and so is vertex decomposable. Otherwise, suppose $|\Sigma| > 4$. Let T be a spanning tree for Σ, and choose v_0 any terminal vertex of T. Then $\Sigma\backslash v_0$ contains $T\backslash v_0$ which is a spanning tree for $\Sigma\backslash v_0$, hence $\Sigma\backslash v_0$ is 2-dimensional and connected with $|\Sigma\backslash v_0| < |\Sigma|$. Therefore $\Sigma\backslash v_0$ is vertex decomposable, and $\ell k_{v_0} \Sigma$ is vertex decomposable by Theorem 4.1.1. Hence v_0 is a shedding simplex of Σ, and therefore Σ is vertex decomposable.

Theorem 4.1.3: 2-spheres and 2-balls (simplicial complexes whose realizations are homeomorphic to S^2 or B^2) are vertex decomposable. Hence the dual complex of a simple 3-polyhedron is vertex decomposable.

Proof: We refer the reader to Chapter 5 for clarification of terms and also for the proofs of several facts which, by their topological nature,

are also relegated to Chapter 5. They are:

1) Simplicial 2-spheres and 2-balls are pure dimensional.

2) If Σ is a 2-sphere or 2-ball, and v is any vertex in Σ, then $\ell k_{\Sigma} v$ is either a 1-sphere or a 1-ball, which corresponds to a graph which is a single open or closed non-intersecting path.

3) If Σ is a 2-sphere, and v is any vertex in Σ, then $\Sigma \backslash v$ is a 2-ball.

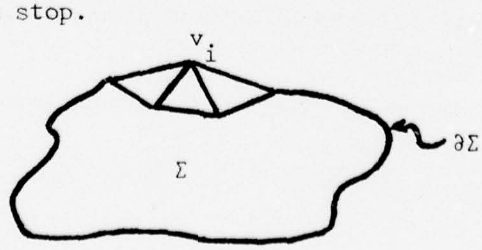
4) If Σ is a 2-ball or 2-sphere and v is any vertex in Σ, then $\Sigma \backslash v$ is a 2-ball iff $\ell k_{\Sigma} v \cap \partial\Sigma = \partial \ell k_{\Sigma} v$.

5) The dual complex of a polyhedron is a ball or sphere.

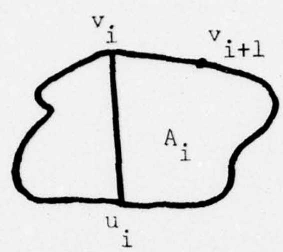
We prove the theorem by showing that every 2-sphere and every 2-ball with more than one 2-simplex contains a vertex v for which $\Sigma \backslash v$ is a 2-ball. Further, from (2) we have $\ell k_{\Sigma} v$ is connected and hence vertex decomposable by Theorem 4.1.2. Hence we have, by an induction argument, that v is a shedding vertex for Σ, and so Σ is vertex decomposable.

If Σ is a sphere, then from (3) any vertex in Σ can be removed to form a 2-ball. Assume then, that Σ is a 2-ball, so that Σ can be placed on to the plane as a graph with an unbounded region and triangular interior regions. Choose v_0 on the boundary of Σ and proceed as follows (assuming vertex v_i has been defined):

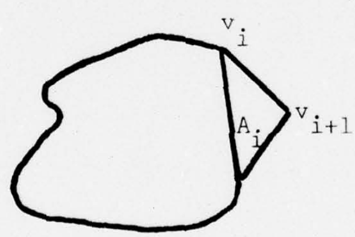
1) If no edge of Σ containing v_i cuts entirely through the center of Σ, then stop.



2) Otherwise, let $\{v_i, u_i\}$ be the cutting edge, and continue clockwise around the boundary of Σ to adjacent vertex v_{i+1}. Go to 1.



First we show this procedure stops. Let A_i be that part of Σ to the "right" of $\{v_i, u_i\}$. Then A_i must contain at least one triangle (since $\{v_i, u_i\}$ cuts through the interior of Σ), and A_{i+1} is strictly contained in A_i (since u_{i+1} is to the "right" of u_i, and v_{i+1} is strictly to the right of v_i). So A_i must eventually be a single triangle,



and so v_{i+1} stops the procedure.

What we get from the procedure is some v for which the only vertices u adjacent to v which are on the boundary of Σ are those for which $\{v,u\}$ is also on the boundary. Hence

$$\ell k_\Sigma v \cap \partial\Sigma = \partial(\ell k_\Sigma v)$$

and so $\Sigma \backslash v$ is a 2-ball. This completes the proof of Theorem 4.1.3.

To end the chapter, we comment that by exhaustive search through the complexes in [18], it seems that all 3-spheres with at most 8 vertices are vertex decomposable. We will not enumerate the decompositions here.

4.2 Complexes on Matroids

A *matroid* $M = (E,\mathcal{I})$ consists of a finite set E together with a non-empty collection $\mathcal{I}$ of subsets of E, called *independent sets*, with the properties 1) every subset of an independent set is independent, and 2) for each set $A \subseteq E$, the elements of $\mathcal{I}$ which are maximal with respect to being contained in A all have the same cardinality $r(\mathcal{I},A)$. Then by (1), $\mathcal{I}$ forms a simplicial complex on E, and by (2), $\mathcal{I}$ is pure $(r(\mathcal{I},E)-1)$-dimensional. (For more information on matroids see [39],[37].)

Theorem 4.2.1: If $\mathcal{I}$ is the collection of independent sets of a matroid M on E, then $\mathcal{I}$ is vertex decomposable.

Proof: We prove that $\mathcal{I}$ satisfies Definition 1. If $|\mathcal{I}| = 1$ then $\mathcal{I} = \{\emptyset\}$ which is vertex decomposable. Otherwise proceed by induction on $|\mathcal{I}| > 1$. Choose any vertex v in $\mathcal{I}$. Then

$$\begin{aligned} \mathcal{I}\setminus v &= \{\tau \in \mathcal{I} \mid v \notin \tau\} \\ &= \{\tau \in \mathcal{I} \ \tau \subseteq E\setminus v\} \end{aligned}$$

which satisfies (1) and (2) $(r(\mathcal{I}\setminus v,A) = r(\mathcal{I},A))$, and hence is a matroid

(called the "deletion (matroid) of v from M"), with $|I\backslash v| < |I|$. *Hence by induction* $I\backslash v$ *is vertex* decomposable. Also

$$\ell k_I v = \{\tau \in I \mid v \notin \tau,\ \tau \cup \{v\} \in I\}$$

which again satisfies (1), and satisfies (2) since, for every $A \subseteq E\backslash v$ and every set $\tau \in I$ with $v \in \tau \subseteq A \cup \{v\}$, τ is contained in some maximal element of I contained in A, which also contains v, hence $r(\ell k_I v, A) = r(I, A) - 1$. Therefore $\ell k_I v$ is a matroid (called the "contraction of v in M") with $|\ell k_I v| < |I|$, and so by induction $\ell k_I v$ is also vertex decomposable.

Thus v is a shedding vertex for I, and so I itself is vertex decomposable.

Corollary 4.2.2: Matroids are shellable and satisfy the Hirsch Conjecture.

We call the circuits of a matroid M that collection $C(I)$ of minimal sets in E which are not contained in I. Given an ordering $e_1, \ldots, e_m$ of the elements of E, we call the broken circuits of (with respect to the ordering) the collection of sets $C\backslash e_k$, $C \in C(I)$, e_k that element of C with highest index. Finally we define the broken circuit complex on E to be

$$B(I) = \{\tau \subseteq E \mid \tau \text{ contains no broken circuit}\}.$$

We state without proof two propositions of Brylawski.

Proposition 4.2.3: If M is a matroid on E with $r(\mathcal{I},E) = d$, then $\mathcal{B}(\mathcal{I})$ is a pure (d-1)-dimensional complex ([9] Proposition 3.1).

Proposition 4.2.4: Let $M = (\mathcal{I}, E = \{e_1,\ldots,e_m\})$ be a matroid, $\mathcal{B}(\mathcal{I})$ its associated broken circuit complex. If any element of E is not a vertex of $\mathcal{I}$, then $\mathcal{B}(\mathcal{I}) = \emptyset$. Otherwise,

$$\mathcal{B}(\mathcal{I})\backslash e_m = \mathcal{B}(\mathcal{I}\backslash e_m)$$

$$\ell k_{\mathcal{B}(\mathcal{I})} e_m = \mathcal{B}(\ell k_{\mathcal{I}} e_m).$$

([9] Proposition 3.2 a,d,e).

With these facts we prove

Theorem 4.2.5: Non-empty broken circuit complexes are strongly vertex decomposable.

Proof: Let $M = (\mathcal{I}, E = \{e_1,\ldots,e_m\})$ be a matroid $\mathcal{B}(M)$ its associated broken circuit complex. We show that $\mathcal{B}(\mathcal{I})$ satisfies Definition 1. $\mathcal{B}(\mathcal{I})$ is pure dimensional by Proposition 4.2.3 $\mathcal{I} = 1$ implies $\mathcal{I} = \{\emptyset\}$, and so $C(\mathcal{I}) = \{\{e\} | e \in E\}$. Therefore $\mathcal{B}(\mathcal{I}) = \{\emptyset\}$, which is vertex decomposable.

For $|\mathcal{I}| > 1$, proceed by induction on $|\mathcal{I}|$. We have, by Proposition 4.2.4, $\mathcal{B}(\mathcal{I}) \neq \emptyset$ implies every element of E is a vertex of $\mathcal{I}$, and so

$$\mathcal{B}(\mathcal{I})\backslash e_m = \mathcal{B}(\mathcal{I}\backslash e_m)$$

$$\ell k_{\mathcal{B}(\mathcal{I})} e_m = \mathcal{B}(\ell k_{\mathcal{I}} e_m).$$

But these are both non-empty broken circuit complexes on the matroid independent sets $\mathcal{I}\setminus e_m$, $\ell k_{\mathcal{I}} e_m$ (see Theorem 4.2.1) with $|\mathcal{I}\setminus e_m| < |\mathcal{I}|$, $|\ell k_{\mathcal{I}} e_m| < |\mathcal{I}|$. Hence by induction they are both vertex decomposable. Therefore e_m is a shedding simplex of $\mathcal{B}(\mathcal{I})$, and so $\mathcal{B}(\mathcal{I})$ is vertex decomposable.

Corollary 4.2.6: Broken circuit complexes are shellable and satisfy the Hirsch Conjecture.

4.3 Leontief Complexes

A pure (d-1)-dimensional complex Σ *on set* E is called Leontief if 1) for all $\tau \in \Sigma$ (including $\tau = \emptyset$), $\ell k_{\Sigma}\tau$ is simplicially path *connected, that is, every two maximal simplices in* $\ell k_{\Sigma}\tau$ are connected by a simplicial path in $\ell k_{\Sigma}\tau$, and 2) there is a labelling of the m elements of E by the numbers $1,\ldots,m-d$ so that for every $(d-1)$-simplex σ in Σ, $E\setminus\sigma$ has a complete labelling, that is, exactly one each of the numbers 1 through $m-d$.

Leontief complexes are generalizations of the dual complexes Σ_P^* to simple Leontief substitution systems, i.e. simple polyhedra of the form

$$P = \{x \in \mathbf{R}^m | Ax = b, \quad x \geq 0\}$$

where b is a non-negative α vector and A is a $d\times m$ Leontief matrix, that is, A is rank d, A contains exactly one positive element in each column, and $Ax > 0$ has a non-negative solution.

By remarks in Section 2.6 we know that every face of P has a path connected boundary complex, and so from Proposition 2.4.1 we have that the link of every simplex in Σ_P^* is path connected. Further, from the characterizations of Dantzig [12] and Veinott [38] it follows that the extreme points of P can be expressed as $B^{-1}b$ on the columns of B and 0 elsewhere, where B is a square matrix comprised of columns of A with exactly one positive element per row (and hence per column). The assumption that P is simple implies that $\dim P = \dim \Sigma_P^* + 1 = m-d$. We can therefore label the columns of A with the numbers $1,\ldots,d = m-\dim P$ according to which entry of that column is positive. Then the vertices of Σ_P^* correspond to the facets of P, which are among the sets $\{x \in P | x_i = 0\}$ $i = 1,\ldots,m$, and the (d-1)-simplices of Σ_P^* correspond to vertices of P, which are therefore contained in a set of facets whose corresponding set of matrix columns is the complement of a set of columns with a complete labelling. Therefore Σ_P^* is an $m-d-1$ dimensional Leontief complex.

Two facts follow immediately from the properties of the labelling, namely:

1) All (d-1)-simplices in Σ have the same (not necessarily distinct or complete) set of labels, since their complements in E do.

2) If m is the number of vertices in Σ, then $m \leq 2d$, since if $m-d > d$, then for some (d-1)-simplex σ, the label of at least one vertex v is exclusively in $E\setminus\sigma$, hence by (1), is exclusively in $E\setminus\sigma$ for every (d-1)-simplex σ, implying the impossible fact that the vertex v is in no (d-1)-simplex.

It is also true that the property of being Leontief is preserved down to links.

Proposition 4.3.1: If Σ is Leontief, $\tau \in \Sigma$, then $\ell k_\Sigma \tau$ is Leontief.

Proof: Certainly $\ell k_{\ell k_\Sigma \tau} \sigma = \ell k_\Sigma (\tau \cup \sigma)$ is path connected for $\sigma \in \ell k_\Sigma \tau$. Let v_τ be the set of vertices in $\ell k_\Sigma \tau$ and let σ be a $(d-1-|\tau|)$-simplex in $\ell k_\Sigma \tau$. Then $v_\tau \backslash \sigma \subseteq v \backslash (\sigma \cup \tau)$, which has a complete labelling, and so $v_\tau \backslash \sigma$ has a distinct set of labels. But every $(d-1-|\tau|)$-simplex σ' in $\ell k_\Sigma \tau$ has the same set of labels as σ, since $\sigma' \cup \tau$ has the same set of labels as $\sigma \cup \tau$. Therefore $v_\tau \backslash \sigma'$ has the same distinct labelling as $v_\tau \backslash \sigma$, and so by relabelling the vertices in $\ell k_\Sigma \tau$ by the numbers $1 \ldots n-d-|\tau|$ we have that $\ell k_\Sigma \tau$ is Leontief.

The first theorem of this section is a generalization of the result of Grinold in [16], and uses the same basic proof.

Theorem 4.3.2: Leontief complexes satisfy the Hirsch Conjecture.

Proof: We prove the following claim which proves the theorem.

Claim: If Δ_1 and Δ_2 are two distinct $(d-1)$-simplices in Σ, then there exists a vertex $v_1 \in \Delta_1 \backslash \Delta_2$, $v_2 \in \Delta_2 \backslash \Delta_1$ so that $(\Delta_1 \backslash \{v_1\}) \cup \{v_2\}$ is a $(d-1)$simplex in Σ.

For, if σ_1 and σ_2 are $(d-1)$-simplices in Σ, then $|\sigma_1 \cup \sigma_2| \leq n$ implying

$$|\sigma_1 \backslash \sigma_2| \leq |\sigma_1 \cup \sigma_2| \backslash \sigma_2 \leq n-d$$

and so by repeated application of the claim we obtain a path from σ_1 to σ_2 of length at most $n-d$.

Proof of Claim: By induction on $|\Sigma|$. If Σ has only one (d-1)-simplex, then the claim is true vacuously. Otherwise, let $|\Sigma| > 1$ and $\Delta_1 \neq \Delta_2$ be (d-1)-simplices in Σ.

Case 1 ($\Delta_1 \cap \Delta_2 = \emptyset$): We have $|\Delta_1 \cup \Delta_2| = 2d$ and so $\Delta_1 \cup \Delta_2$ must comprise all the vertices of Σ. Now let Δ' be the first simplex in any path from Δ_1 to Δ_2 (there must be at least one such path). Then $\Delta_1 \backslash \Delta' = \{v_1\}$, $\Delta' \backslash \Delta_1 = \{v_2\}$ and v_2, since it is not in Δ_1, must be in Δ_2. Therefore $\Delta' = (\Delta_1 \backslash \{v_1\}) \cup \{v_2\}$ is a (d-1)-simplex in Σ.

Case 2 ($\Delta_1 \cap \Delta_2 = \tau \neq \emptyset$): We have $\Delta_1 \backslash \tau$ and $\Delta_2 \backslash \tau$ are $(d-1-|\tau|)$-simplices in $\ell k_\Sigma \tau$, which is Leontief by Proposition 4.3.1 and has $|\ell k_\Sigma \tau| < |\Sigma|$. Hence by induction, there are vertices $v_1 \in (\Delta_1 \backslash \tau) \backslash (\Delta_2 \backslash \tau) = (\Delta_1 \backslash \Delta_2) \backslash \tau$, $v_2 \in (\Delta_2 \backslash \tau) \backslash (\Delta_1 \backslash \tau) = (\Delta_2 \backslash \Delta_1) \backslash \tau$ so that $((\Delta_1 \backslash \tau) \backslash \{v_1\}) \cup \{v_2\}$ is a $d-1-|\tau|$ simplex in $\ell k_\Sigma \tau$. Therefore $(\Delta_1 \backslash \{v_1\}) \cup \{v_2\} = [((\Delta_1 \backslash \tau) \backslash \{v_1\}) \cup \{v_2\}] \cup \tau$ is a (d-1)-simplex in Σ, and so the claim, and hence the theory, is proved.

For the vertex decomposability result we need to restrict ourselves to "bounded" Leontief complexes.

Definition: A pure (d-1)-dimensional simplicial complex Σ is called bounded if every (d-2)-simplex of Σ is contained in at least two (d-1)-simplices of Σ, i.e. $\partial \Sigma = \emptyset$.

Thus the simplicial duals of bounded polyhedra are bounded complexes, since every edge of a polytope contains two vertices, whereas the simplicial duals of unbounded polyhedra are not bounded complexes, since

unbounded polyhedra contain unbounded edges (edges containing only one vertex).

We note that links in bounded complexes are also bounded for if $\tau \in \Sigma$ and σ is a $(d-|\tau|-2)$-simplex in $\ell k_\Sigma \tau$, then $\sigma \in \tau$ is a $(d-\Sigma)$-simplex in Σ, and so is contained in at least two (d-1)-simplices σ_1, σ_2 in Σ. But then σ_1 and σ_2 both contain σ, and so $\sigma_1 \backslash \tau$, $\sigma_2 \backslash \tau$ are $(k-|\tau|-1)$-simplices in $\ell k_\Sigma \tau$ containing σ.

Theorem 4.3.3: Bounded Leontief complexes form the independent sets of a matroid.

Proof: Let Σ be a bounded Leontief complex of dimension $d-1$ on set E. Certainly Σ is closed under inclusion. Now let $A \subseteq E$ and σ_1, σ_2 be two elements of Σ which are maximal with respect to being contained in A. Then it must be true that $\sigma_1 = \Delta_1 \cap A$, $\sigma_2 = \Delta_2 \cap A$, where Δ_1, Δ_2 are (d-1)-simplices in Σ, in particular $E \backslash \Delta_1$, $E \backslash \Delta_2$ are completely labelled.

Now suppose $|\sigma_1| > |\sigma_2|$. Then $|A \backslash \sigma_1| < |A \backslash \sigma_2|$, and both these sets have distinct labels. Hence there must be some label in $A \backslash \sigma_2$ which is not in $A \backslash \sigma_1$. Let v be the vertex with this label. There must also be a vertex v' in $E \backslash \Delta_1$ with the same label as v, and so of course $v' \in \Delta_2$. Now Σ bounded implies that there is a (d-1)-simplex $\Delta' \neq \Delta_2$ in Σ containing $\Delta_2 \backslash \{v'\}$. But Δ' has the same set of labels as Δ_2, and the only other vertex in $E \backslash (\Delta_2 \backslash \{v'\})$ with that label must be v. So

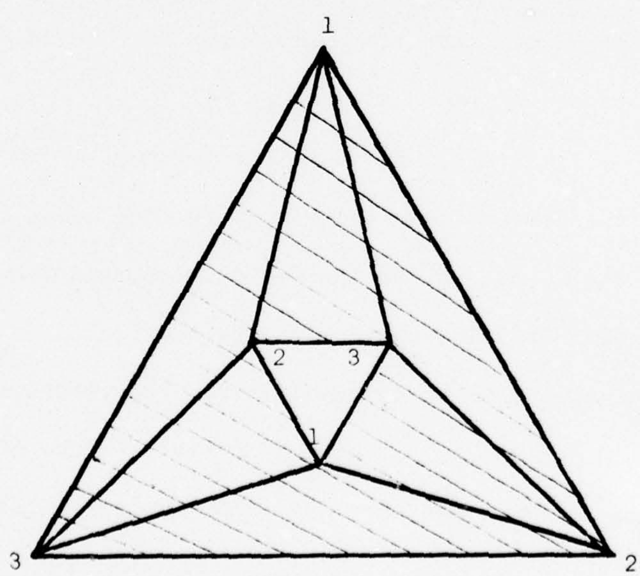
$$\Delta' \cap A = [(\Delta_2 \setminus \{v'\}) \cup \{v\}] \cap A$$

$$= \sigma_2 \cup \{v\}$$

implying that σ_2 was not maximal as a simplex of Σ contained in A. Hence $|\sigma_2| = |\sigma_1|$ and so all elements of Σ which are maximal with respect to being contained in A have the same cardinality. Therefore Σ forms the independent sets of a matroid.

Corollary 4.3.4: Bounded Leontief complexes are vertex decomposable, and hence also shellable.

Note: Boundedness is essential here, for the complex below is a Leontief complex, appropriately labelled, which is not vertex decomposable (removal of any vertex leaves a complex which is not pure).



4.4 Distributive Lattice Complexes

Let $P = (E,\leq)$ be a finite partially ordered set. We say that P is a lattice if every two elements a,b in P have a least upper bound $a \vee b$ - an element $y \in P$ with $y \geq a$, $y \geq b$, and $y \leq x$ for every $x \in P$ with $x \geq a$, $x \geq b$ - and a greatest lower bound $a \wedge b$ - an element $y \in P$ with $y \leq a$, $y \leq b$, and $y \geq x$ for every $x \in P$ with $x \leq a$, $x \leq b$. P is a distributive lattice, if in addition the operations $\wedge$, $\vee$ satisfy either of the two equivalent properties

1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad a,b,c \in P$

2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad a,b,c \in P.$

An important property of distributive lattices (see [7] III.3) is stated here without proof. Let L be a distributive lattice, and define a meet irreducible element of L to be any element x of L with exactly one successor, that is, there exists a unique $y \in L$ such that $y \geq x$ and if $y \geq z \geq x$, then $z = y$ or $z = x$. These elements form a partially ordered set under $\leq$, which we shall denote P_L. Define an order ideal of P_L to be any subset I of P_L closed under $\leq$, that is, if $x \in I$ and $y \leq x$, then $y \in I$. Then:

Proposition 4.4.1: L is isomorphic as a lattice to the set of order ideals of P_L with $\leq$ being replaced by $\subseteq$, $\wedge$ by $\cap$, and $\vee$ by $\cup$.

This means that every maximal chain (totally ordered set) of L has the same cardinality $n+1$, where n is the number of irreducible

elements of L. So the lattice complex of L, Σ_L, defined on the elements of L to be $\Sigma_L = \{S \subseteq L | S$ is a chain$\}$, is a pure n-dimensional simplicial complex. We will spend the remainder of this section proving the following.

<u>Theorem 4.4.2</u>: If L is a distributive lattice, and Σ_L is of dimension d, then Σ_L can be derived from the d-simplex by a series of stellar subdivisions.

By Proposition 3.2.3 we have an immediate corollary.

<u>Corollary 4.4.3</u>: If L is a distributive lattice, then Σ_L is vertex decomposable, hence is shellable and satisfies the Hirsch conjecture.

We first prove a lemma.

<u>Lemma</u>: If $\Sigma = \bar{v}.\Sigma'$ and $(a_1,X_1),\ldots,(a_n,X_n)$ is a series of stellar subdivisions performed on Σ $(\Sigma_0 = \Sigma,\ \Sigma_i = st(a_i,X_i)[\Sigma_{i-1}])$ such that $v \notin X_i$ $i = 1,\ldots,n$, then the resulting complex is identical to the complex $\bar{v}.\Sigma''$, where Σ'' is obtained from Σ' by the same series of stellar subdivisions.

<u>Proof</u>: It is sufficient to prove

$$st(a,X_1)[\Sigma] = \bar{v}.st(a_1,X_1)[\Sigma'].$$

But $v \notin X_1$ implies $X_1 \in \Sigma'$, and so

$$\begin{aligned} st(a_1,X_1)[\Sigma] &= (\Sigma\backslash X_1) \cup \bar{a}_1.\partial X_1.\ell k_\Sigma X_1 \\ &= \bar{v}.(\Sigma'\backslash X_1) \cup \bar{a}_1.\partial X.\bar{v}.\ell k_{\Sigma}.X_1 \quad \text{(Lemma 2.3.2)} \\ &= \bar{v}.st(a_1,X_1)[\Sigma]. \end{aligned}$$

Proof of Theorem: Let $P_L = (E,\leq)$, so that Σ_L has vertices labelled v_S, where $S \subseteq E$ is an order ideal of P_L, and $\dim \Sigma_L = |E|$. We prove the following claim, which proves the theorem:

Claim: Let $\Delta = \{v_{\langle a\rangle} | \langle a\rangle = \{x \in E | x \leq a\}, a \in E\} \cup \{v_{\{\emptyset\}}\}$, and let $S_1,\ldots,S_n$ be any ordering of the order ideals of P_L such that $|S_1| \geq |S_2| \geq \ldots \geq |S_n|$. Finally let $X_i = \{v_{\langle \alpha\rangle}|a$ is a maximal element in $S_i\}$. Then $(v_{S_1},X_1),\ldots,(v_{S_n},X_n)$ is a series of stellar subdivisions which, when performed on the complex $\bar{\Delta}$, yields Σ_L.

Proof of Claim: The proof is by induction on $|E|$. If $|E| = 1$ then $\Sigma_L = \{v_E,v_{\{\emptyset\}}\}$, $\Delta = \{v_E,v_{\{\emptyset\}}\}$, and $X_1 = \{v_E\}$. Hence $st(v_E,X_1)[\bar{\Delta}]$ is again $\bar{\Delta} = \Sigma_L$. Assume, then, that $|E| > 1$. We have $v_E = v_{S_1}$ is a vertex in Σ_L which is in every maximal simplex of Σ_L, since E is in every maximal chain of L. Thus $\Sigma_L = \bar{v}_E.\ell k_{\Sigma_L} v_E$. Further, since $S_2,\ldots,S_n$ are proper subsets of E, then $X_2,\ldots,X_n$ cannot contain V_E. Hence it remains to prove, by the lemma, that $\ell k_{\Sigma_L} v_E$ is the stellar subdivision of $\ell k_{st(v_E,X_1)[\bar{\Delta}]} v_E$ by $(v_{S_2},X_2),\ldots,(v_{S_n},X_n)$.

Now,

$$\ell k_{st(v_E, X_1)[\bar{\Delta}]} = \ell k_{\bar{v}_E} . \partial X_1 . \ell k_{\bar{\Delta}} X_1 \, v_E \quad \text{(Lemma 2.3.1)}$$

$$= \partial X_1 . \ell k_{\bar{\Delta}} X_1 \quad \text{(Lemma 2.3.2)}$$

$$= cl\{(X_1 \backslash v_{\langle a\rangle}).(\Delta \backslash X_1) | a \text{ is a maximal element in } E\ (=S_1)\}$$

$$= cl\{\Delta \backslash v_{\langle a\rangle} | a \text{ is a maximal element in } E\}$$

which is non-empty, since $|\Delta| = |E|+1 > 0$.

For each $\Delta\backslash v_{\langle a\rangle}$ as above, associate the complex Σ_a of $\ell k_{\Sigma_L} v_E$ generated by the (d-1)-simplices $\{\sigma_\Gamma | \Gamma \cup \{E\}$ is a maximal chain in L with $E \notin \Gamma$ and $E\backslash\{a\} \in \Gamma\}$. Then the Σ_a's partition the (d-1)-simplices of $\ell k_{\Sigma_L} v_E$, since each chain contains exactly one set of size $|E|-1$. *Now each* Σ_a is *itself a* distributive lattice complex of the form Σ_{L_a}, where $P_{L_a} = (E\backslash\{a\}, \leq)$, since $v_{u_1} \ldots v_{u_k} \in \Sigma_a$ iff $\{u_1, \ldots, u_k\}$ is the subset of some maximal chain Γ in L with $E \notin \Gamma$ and $E\backslash\{a\} \in \Gamma$ iff $\{u_1, \ldots, u_k\}$ is the subset of some maximal chain in L_a iff $v_{u_1} \ldots v_{u_k} \in \Sigma_{L_a}$. Finally, the set

$$\Delta_a = \{v_{S_b} | S_b = \{x \in E\backslash\{a\} | x \leq b\},\ b \in E\backslash\{a\}\}$$

$$= \{v_{\langle b\rangle} | b \in E\backslash\{a\}\} \quad \text{(since } a \text{ is maximal)}$$

$$= \Delta \backslash v_{\langle a\rangle}.$$

Hence by induction on $|E\backslash\{a\}| < |E|$, we have, for the ordering $S_{j_1}, \ldots, S_{j_p}$ of those S_j which are order ideals of P_{L_a},

$j_1 < \ldots < j_p$, that $(v_{S_{j_1}}, X^a_{j_1}), \ldots, (v_{S_{j_p}}, X^a_{j_p})$ is a series of stellar subdivisions of $\bar{\Lambda}_a$ which yields Σ_a, where

$$X^a_{j_k} = \{v_{S_b} | S_b \in \Delta_a \text{ with } b \text{ a maximal element in } S_{j_k}\}$$

$$= \{v_{\langle b \rangle} | b \text{ a maximal element in } S_{j_k}\}$$

(again since a is maximal)

$$= X_{j_k} .$$

Therefore $(v_{S_2}, X_2), \ldots, (v_{S_n}, X_n)$ is a series of stellar subdivisions of

$$\cup \{\Sigma_a | a \text{ a maximal element of } E\} = \ell k_{st(v_E, X_1)[\bar{\Delta}]} v_E$$

which yields $\ell k_{\Sigma_L} v_E$. This proves the claim and hence the theorem.

Note: Sections 4.2 and 4.4 deal with three classes of complexes of a type known as "constructible" complexes (see [36]). Stanley [34] cites these classes, along with the boundary complexes of polytopes, as the four known classes of constructible complexes, and poses the question: "Are constructible complexes shellable?" Of course the fourth class is shellable by Proposition 2.5.1, and now the first three classes have been established as shellable classes.

4.5 The Boundary Complex of a Cyclic Polytope

The cyclic polytope $C(n,d)$ of dimension $d \geq 1$ with $n \geq d+1$ vertices ([17] §4.7) defined to be the convex hull of the points

$$\{v_i = (t_i, t_i^2, \ldots, t_i^d) \in R^d\}_{i=1}^n$$

where t_i are real numbers with $t_1 < t_2 \ldots < t_n$. $C(n,d)$ is a simplicial polytope, and so has an associated (d-1)-dimensional boundary simplicial complex, which we will also call $C(n,d)$, whose maximal simplices can be described on the set $V = \{v_1, \ldots, v_n\}$ as follows:

> Gale's Evenness Criterion (GEC): A d-set $\sigma \subseteq v$ is a (d-1)-simplex of $C(n,d)$ iff each two vertices in $v \backslash \sigma$ are separated (in the ordering $v_1, \ldots, v_n$) by an even number of vertices of σ.

Table 1 displays the example $C(8,3)$, where the 3-simplices are given by the rows. Cyclic polytopes and their complexes are important because the face structure maximizes simultaneously for all k the number of k-faces in a d-polytope (or even simplicial (d-1)-sphere) with n vertices (see [27] and [36]). They have been further studied, with respect to their simplicial diameter, in [21].

Theorem 4.5.1: The boundary complex of a cyclic polytope is vertex decomposable.

We first prove a lemma:

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	1	1	1				
1	1	1					1
1	1		1	1			
1	1			1	1		
1	1				1	1	
1	1					1	1
1		1	1				1
1			1	1			1
1				1	1		1
1					1	1	1
	1	1	1	1			
	1	1		1	1		
	1	1			1	1	
	1	1				1	1
		1	1	1	1		
		1	1		1	1	
		1	1			1	1
			1	1	1	1	
			1	1		1	1
				1	1	1	1

Table 2

C(8,3)

Lemma: Let the simplicial complex $C(n,d)$ be described on vertex set $\{v_1,\ldots,v_n\}$ by Gale's Evenness Criterion. Then:

1) For $k \leq n-d-1$, $C(n,d)\backslash v_n\backslash\ldots\backslash v_{n-k}$ is pure $(d-1)$-dimensional.

2) For $n \geq d+2$, $C(n,d)\backslash v_n\backslash v_{n-1} = C(n-1,d)\backslash v_{n-1}$.

3) For $d \geq 2$, $\ell k_{C(n,d)} v_n = C(n-1,d-1)$.

4) For $d \geq 3$, $\ell k_{C(n,d)\backslash v_n} v_{n-1} = \bar{v}_{n-2}.[C(n-2,d-2)\backslash v_{n-2}]$.

Proof of lemma: 1) Let $\tau \in C(n,d)\backslash v_n\backslash\ldots\backslash v_{n-k}$, so that $v_{n-i} \notin \tau$, $i = 0,\ldots,k$, and $\tau \subseteq \sigma$ for σ of dimension $d-1$ satisfying GEC. If $\sigma \cap \{v_{n-k},\ldots,v_n\} = \emptyset$, then we are done. Otherwise, we can delete the contiguous blocks of vertices of σ containing $v_n,\ldots,v_{n-k}$ and redistribute those vertices into the lowest indices not included in σ. The new d-set σ' again satisfies GEC, since any two vertices in $\{v_1,\ldots,v_{n-i-1}\}\backslash\sigma'$ are separated by the same set of vertices as they were in $\{v_1,\ldots,v_n\}\backslash\sigma$. Therefore σ' is a $(d-1)$-simplex in $C(n,d)\backslash v_n\backslash\ldots\backslash v_{n-k}$, and so it follows that $C(n,d)\backslash v_n\backslash\ldots\backslash v_{n-k}$ is pure $(d-1)$-dimensional.

2) By (1), we need only prove that the $(d-1)$-simplices of $C(n,d)\backslash v_n\backslash v_{n-1}$ are the same as those in $C(n-1,d)\backslash v_{n-1}$. But a d-set $\sigma \subseteq \{v_1,\ldots,v_{n-2}\}$ is a $(d-1)$-simplex in $C(n,d)\backslash v_n\backslash v_{n-1}$ iff $\sigma = \{v_1,\ldots,v_d\}$ as σ satisfies GEC on $\{v_1,\ldots,v_n\}$ and there are an even number of vertices in the last block iff σ satisfies GEC on $\{v_1,\ldots,v_{n-1}\}$ and there are an even number of vertices in the last block iff σ is a $(d-1)$-simplex of $C(n-1,d)\backslash v_{n-1}$.

3) $\tau \subseteq \{v_1,\ldots,v_{n-1}\}$ is a $(d-2)$-simplex in the (pure) complex $\ell k_{C(n,d)} v_n$ iff $\sigma = \tau \cup \{v_n\}$ is a $(d-1)$-simplex in $C(n,d)$ iff

$v_n \in \sigma$ and σ satisfies GEC on $\{v_1,\dots,v_n\}$ if $v_n \in \sigma$ and $\sigma \backslash v_n$ satisfies GEC on $\{v_1,\dots,v_{n-1}\}$ iff $\tau \in C(n-1,d-1)$.

4) $C(n,d)\backslash v_n$ is pure dimensional by part (1). Further, $\tau \subseteq \{v_1 \dots v_{n-2}\}$ is a (d-2)-simplex in $\ell k_{C(n,d)\backslash v_n} v_{n-1}$ iff $\sigma = \tau \cup \{v_{n-1}\}$ is a (d-1)-simplex in $C(n,d)\backslash v_n$ iff $v_{n-1} \in \sigma \subseteq \{v_1 \dots v_{n-1}\}$ and σ satisfies GEC on $\{v_1,\dots,v_{n-1}\}$ and has an <u>even</u> number of vertices in the last block iff $\tau = \sigma\backslash\{v_{n-1}\}$ satisfies GEC on $\{v_1,\dots,v_{n-2}\}$ and has an <u>odd</u> number of vertices in the last block iff $v_{n-2} \in \tau$ and $\tau\backslash v_{n-2}$ is a (d-2)-set which satisfies GEC on $\{v_1 \dots v_{n-3}\}$ and has an <u>even</u> number of vertices in the last block iff τ is a (d-2)-simplex in $v_{n-2}\cdot[C(n-2,d-2)\backslash v_{n-2}]$. And so, since $\ell k_{C(n,d)\backslash v_n} v_{n-1}$ is defined by its (d-2)-simplices, then (4) follows.

<u>Proof of Theorem</u>: By definition $C(n,d)$ is pure (d-1)-dimensional. We prove by induction on $|C(n,d)|$ that v_n is a shedding simplex on $C(n,d)$ by Definition 2.

First note that $C(n,1) = cl\{v_1,v_n\}$ (a two point complex) and so clearly v_n is a shedding simplex for $C(n,1)$. This gives us the statement for $|C(n,d)| = 3$.

Now let $|C(n,d)| > 3$. We have by (3) of the lemma that $\ell k_{C(n,d)} v_n = C(n-1,d-1)$, and so is vertex decomposable by induction.

Case 1 $(n = d+1)$: Then $C(n,d)\backslash v_n$ is a (d-1)-simplex and so is vertex decomposable.

Case 2 $(n \geq d+2)$: By (1) of the Lemma $C(n,d)\backslash v_n$ is pure (d-1)-dimensional. Now consider the vertex v_{n-1} in $C(n,d)\backslash v_n$. We have $(C(n,d)\backslash v_n)\backslash v_{n-1} = C(n-1,d)\backslash v_{n-1}$ which is vertex decomposable, since

by induction v_{n-1} is a shedding vertex for $C(n-1,d)$. And $\ell k_{C(n,d)\setminus v_n} v_{n-1}$ is either a point set if $d = 2$, or equals $\bar{v}_{n-2} \cdot [C(n-2,d-2)\setminus v_{n-2}]$ if $d \geq 3$, by (4) of the lemma. In the first case every point set is vertex decomposable and in the second case $C(n-2,d-2)\setminus v_{n-2}$ is vertex decomposable since, by induction on n, v_{n-2} is a shedding vertex for $C(n-2,d-2)$, and so by Proposition 3.2.2, $\bar{v}_{n-2} \cdot \ell k_{C(n,d)\setminus v_n} v_{n-1}$ is vertex decomposable.

Therefore v_{n-1} is a shedding simplex for $C(n,d)\setminus v_n$, and so v_n is a shedding simplex for $C(n,d)$. This completes the induction step, and hence the proof of the theorem.

<u>Corollary 4.5.3</u>. $C(n,d)$ is shellable and satisfies the Hirsch Conjecture.

<u>4.5 Three Non k-decomposable Complexes</u>

To end this chapter, it is only fair to give examples of some fairly reasonable complexes which are not k-decomposable or weakly k-decomposable for various k.

<u>Example 4.5.1</u>: The Klee-Walkup counterexample.

This is the dual complex to the unbounded polyhedron constructed in [24] with presentation

$$\begin{pmatrix} -6 & -3 & 0 & 1 \\ -3 & -6 & 1 & 0 \\ -35 & -45 & 6 & 3 \\ -45 & -35 & 3 & 6 \end{pmatrix} \leq \begin{pmatrix} -1 \\ -1 \\ -8 \\ -8 \end{pmatrix}$$

and dual complex whose 3-simplices are given as the rows in Table 3.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	1	1	1				
1		1	1		1		
1			1	1			1
1			1		1	1	
1			1			1	1
1				1		1	1
	1	1	1	1			
	1	1		1			1
	1	1			1	1	
	1	1				1	1
	1				1	1	1
		1	1	1			1
		1	1		1	1	
		1	1			1	1
				1	1	1	1

Table 3

Dual complex to the Klee-Walkup counterexample

It has dimension 3, 8 vertices, 15 3-simplices and diam $\Sigma = 5 > 8-4$, and hence by Corollary 3.5.2 cannot be strongly vertex decomposable. It is, however, both weakly vertex decomposable through shedding order v_1, v_4, v_5, v_3, and 1-decomposable by v_{12}, v_{24}, v_5, v_8.

Example 4.5.2: The Rudin Counterexample.

This is a triangulation of the geometric 3-simplex constructed in [31] with complex given in Table 4. It has dimension 3, 14 vertices, 41 simplices, and is not shellable, therefore not strongly k-decomposable for any k. The status of weak decomposition is not known.

Example 4.5.3: This is part of a construction by Barnette [5], and is one of the simplest examples of a non vertex decomposable complex. It is a combinatorial ball which is not the dual complex of a polyhedron, since the suspension of its boundary to a point produces Barnette's example, which is not the complex of a polytope (see Chapter 5). Its construction is straightforward. Consider the triangular prism as in Figure 1 on vertices a, b, c, a', b', and c'. "Twist" triangle abc so that the sides of the prism are no longer planar. The convex hull of {a,a',b,b'}, {a,a',c,c'}, {b,b',c,c'} are three 3-simplices in (see Figure 2). Now these simplices each have two 2-faces which face the inside of the prism, and these faces, along with the 2-simplices abc and a'b'c' form a 2-sphere. Place a point d in the interior of this 2-sphere and let the remaining simplices in Σ be the convex hull of d and these two simplices (see Figure 3).

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
1	1			1				1					
	1	1			1				1				
		1	1			1				1			
1			1				1				1		
1	1			1					1				
	1	1			1					1			
		1	1			1					1		
1			1				1	1					
	1			1				1	1				
		1			1				1	1			
			1			1				1	1		
1							1	1			1		
	1				1			1	1				
		1				1			1	1			
			1				1			1	1		
1				1				1			1		
				1				1	1	1			
					1				1	1	1		
						1		1		1	1		
							1	1	1		1		
1				1					1	1			
	1				1					1	1		
		1				1		1			1		
			1				1	1	1				
1						1			1	1			
	1						1			1	1		
		1		1				1			1		
			1		1			1	1				
1						1				1		1	
	1						1				1		1
		1		1				1				1	
			1		1				1				1
1				1						1		1	
	1				1						1		1
		1				1		1				1	
			1				1		1				1
				1				1		1		1	
					1				1		1		1
						1		1		1		1	
							1		1		1		1
								1	1	1	1		

Table 4

Rudin counterexample

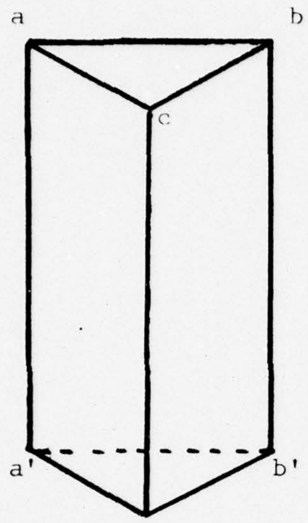


Figure 1

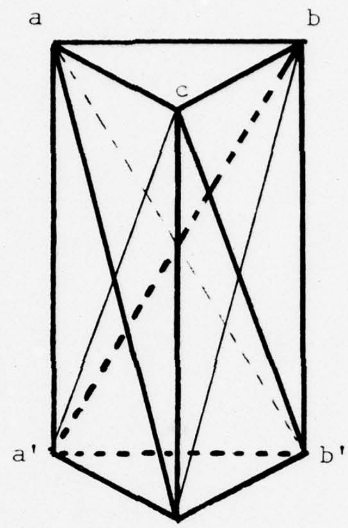


Figure 2

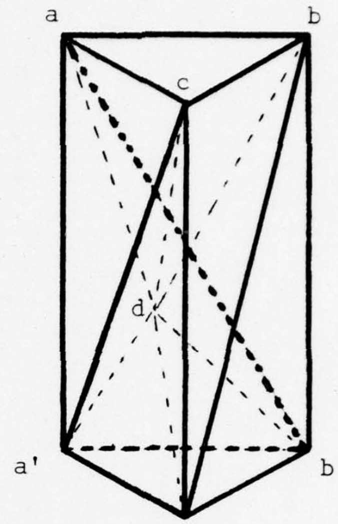


Figure 3

The resulting complex, given in Table 5, has dimension 3, 8 vertices and 11 simplices. It is neither weakly nor strongly vertex decomposable, since removal of any of the points a, b, c, a', b', c' leaves a maximal 2-simplex on the appropriate side simplices, and removal of d leaves a maximal 2-simplex in the top and bottom simplex. It is, however strongly (hence weakly) 1-decomposable by shedding order ac', c, c', a'. It also has diameter $3 \leq n-d-1$, and so even for "reasonable" complexes, the converse to Theorem 3.5.1 is not necessarily true.

Finally, we note that the simplex abcd can be removed from the complex with the same results, and this complex is probably the smallest example of a non-strongly vertex decomposable ball.

a	b	c	a'	b'	c'	d
1	1		1	1		
1		1	1		1	
	1	1		1	1	
1	1	1				1
			1	1	1	1
1	1			1		1
1			1	1		1
	1			1	1	1
	1	1			1	1
1		1	1			1
		1	1		1	1

Table 5

Complex of Example 4.5.3

CHAPTER 5

TOPOLOGICAL PROPERTIES OF k-DECOMPOSITION

5.0 The purpose of this chapter is to investigate the conjecture: All piecewise linear spheres are vertex decomposable. In fact, it is an open question [11] as to whether piecewise linear (PL) spheres are shellable = d-decomposable (PL-balls are not, as Example 4.5.2 shows). We present here what we consider the strongest characterization of strong k-decomposition of piecewise linear spheres which can be obtained by considering only the topological properties of the sphere, and mention two interesting applications to shelling theory. Much of the material in this section is in [26] and [19], so the reader should consult these for clarification or examples.

5.1 Homology Theory

Given a d-dimensional simplicial complex Σ on a set $E = \{e_1, \ldots, e_n\}$, arbitrarily numbered, define for each integer in the m-dimensional simplicial chain group $S_m(\Sigma)$ to be the free abelian group generated by the m-simplices of Σ, i.e., the group with elements of the form

$$a_1\sigma_1 + \ldots + a_p\sigma_p$$

where σ_i are m-simplices, a_i integers. We take the simplex $\emptyset$ to be a (-1)-dimensional simplex, and of course $S_m(\Sigma) = 0$, $m < -1$, $m > d$. The boundary operator $\partial_m : S_m(\Sigma) \to S_{m-1}(\Sigma)$ is defined on **m**-simplex $\sigma = e_{i_1} \ldots e_{i_{m+1}}$ $i_1 < \ldots < i_{m+1}$, to be

$$\partial_m(\sigma) = \sum_{j=1}^{m+1} (-1)^j (\sigma \setminus e_{i_j})$$

and extended linearly to general elements of $S_m(\Sigma)$. It can be shown that $\partial_m \circ \partial_{m+1} = 0$ and so $\mathrm{Im}\, \partial_m \equiv \partial_m(S_m(\Sigma))$ is a subgroup of $\mathrm{Ker}\, \partial_{m-1} \equiv \{\alpha \in S_{m-1}(\Sigma) | \partial(\alpha) = 0\}$, *and so we can define the* <u>reduced homology groups</u>

$$\tilde{H}_m(\Sigma) = \mathrm{Ker}\, \partial_m / \mathrm{Im}\, \partial_{m+1}$$

which turn out to be independent of the numbering of the set E.

The homology theory result of importance to us is the following ([25] Theorem 4.4.6 and subsequent comments).

<u>Theorem 5.1.1</u>: Given complexes L, M, the <u>Mayer-Vietoris sequence</u>, defined for all m by

$$\cdots \to \tilde{H}_m(L \cap M) \xrightarrow{\eta_m} \tilde{H}_m(L) \oplus \tilde{H}_m(M) \xrightarrow{\xi_m} \tilde{H}_m(L \cup M) \xrightarrow{\phi_m} \tilde{H}_{m-1}(L \cap M) \xrightarrow{\eta_{m-1}} \cdots$$

(where η_m, ξ_m and ϕ_m are homomorphisms and where by $A \oplus B$ is meant the group $\{a+b | a \in A, b \in B\}$ with the elements of A and B taken to be distinct), is an exact sequence, that is, $\mathrm{Im}\, \eta_m = \mathrm{Ker}\, \xi_m$, $\mathrm{Im}\, \xi_m = \mathrm{Ker}\, \phi_m$, and $\mathrm{Im}\, \phi_m = \mathrm{Ker}\, \eta_{m-1}$ for all integers m.

For the most part, it is not important to us exactly what the functions η_m, ξ_m, ϕ_m are, only that the sequence is exact. We do, however, need to know η_m for one calculation. For an element $\overline{x}$ of

$\tilde{H}_m(L \cap M)$, which is the equivalence class of an element $x \in S_m(L \cap M) \subseteq S_m(L) \cap S_m(M)$ we define

$$\eta_m(\bar{x}) = (-\bar{x}) \oplus \bar{x} \in \tilde{H}_m(L) \oplus \tilde{H}_m(M).$$

A homology d-manifold is a simplicial complex Σ for which every non-empty simplex σ in Σ has

$$\tilde{H}_m(\ell k_\Sigma \sigma) = \begin{cases} 0 & m \neq d-|\sigma| \\ 0 \text{ or } \mathbb{Z} & m = d-|\sigma| \end{cases}.$$

The boundary of Σ, $\partial\Sigma$, is that set of simplices σ for which $H_{d-|\sigma|}(\ell k_\Sigma \sigma) = 0$. A homology d-manifold is a homology d-ball if $\tilde{H}_m(\Sigma) = 0$ for all m, and a homology d-sphere if

$$\tilde{H}_m(\Sigma) = \begin{cases} 0 & m \neq d \\ \mathbb{Z} & m = d. \end{cases}$$

To relate this definition to the standard one (eg. [26] Definition 5.4.5), and as a useful lemma, we prove

Lemma 5.1.2: If Σ is any simplicial complex, and σ is a non-empty simplex in Σ, then

i) $\tilde{H}_{m-1}((\bar{\sigma}\setminus\sigma).\ell k_\Sigma \sigma) = \tilde{H}_{m-|\sigma|}(\ell k_\Sigma \sigma)$ for all m

ii) $\tilde{H}_m(\bar{\sigma}.\ell k_\Sigma \sigma) = 0$ for all m.

If K is a realization of Σ and $\dot{\sigma}$ is the geometric simplex corresponding to σ in K (see Section 5.2) then $(\bar{\sigma}\backslash\sigma).\ell k_{\Sigma}\sigma$ is the "link" and $\bar{\sigma}.\ell k_{\Sigma}\sigma$ the "closed star," in the sense of [26] Definition 2.4.2, of any point in the relative interior of $\dot{\sigma}$. It becomes a simple argument to show that Σ is a homology d-manifold iff each point in K has a "link" which is a homology ball or sphere, or equivalently, iff every point in K has a "closed star" which is a homology d-ball.

<u>Proof of Lemma</u>: We use a number of specialized topological facts from [26], namely, for Γ any complex:

1) $(\bar{\sigma}\backslash\sigma).\Gamma$ has the same homology as $S_1.S_2. \ldots .S_{|\sigma|-1}.\Gamma$ where $S_i = \{a_i, b_i, \emptyset\}$ are distinct two point complexes (Examples 4.3.12, 2.3.18, and comments at the end of Section 2.3).
2) $\tilde{H}_{m+1}(S_i.\Gamma) = \tilde{H}_m(\Gamma)$ for all m (Theorem 4.4.10).
3) $\tilde{H}_m(\bar{v}.\Gamma) = 0$ for v a vertex not in Γ, all m (Theorem 4.4.10, last sentence of proof).

It follows that

$$\text{i)}\quad \tilde{H}_{m-1}((\bar{\sigma}\backslash\sigma).\ell k_{\Sigma}\sigma) = H_{m-1}(S_1. \ldots .S_{|\sigma|-1}.\ell k_{\Sigma}\sigma)$$

$$= H_{m-2}(S_2. \ldots .S_{|\sigma|-1}.\ell k_{\Sigma}\sigma)$$

$$\ldots = H_{m-|\sigma|}(\ell k_{\Sigma}\sigma) \quad \text{for all } m$$

and

$$\text{ii)}\quad \tilde{H}_m(\bar{\sigma}.\ell k_{\Sigma}\sigma) = \tilde{H}_m(\bar{v}_1.(\overline{\sigma\backslash v_1}).\ell k_{\Sigma}\sigma),\quad v_1 \in \sigma,$$

$$= 0 \quad \text{for all } m.$$

Remarks:

5.1.3: Homology d-manifolds are pure d-dimensional complexes, for if σ is a maximal simplex of Σ of dimension $k < d$, then $\ell k_\Sigma \sigma = \{\emptyset\}$, and so

$$\tilde{H}_{-1}(\ell k_\Sigma \sigma) = \tilde{H}_{-1}(\{\emptyset\}) = \mathbb{Z} \neq 0$$

where $-1 < d-|\sigma|$, a contradiction.

5.1.4: The link of each non-empty simplex in a homology d-manifold is itself a homology sphere or ball, for its links are in turn links of Σ, and hence have the required homology.

5.1.5: The boundary of a homology d-manifold is a homology (d-1)-manifold ([26] *Theorem* 5.4.14), *hence is generated by its* (d-1)-simplices. But a (d-1)-simplex σ is in $\partial\Sigma$ iff $\ell k_\Sigma \sigma$ is a homology 0-ball, i.e. a single point. We have then that $\partial\Sigma$ is exactly those simplices contained in some (d-1)-simplex which is in turn contained in exactly one d-simplex. The simplex $\emptyset$ has not been included in the definition of boundary. Since we will be working exclusively with balls and spheres, however, we can extend the definition to include $\emptyset$, and technically include $\emptyset$ in general manifolds whenever the boundary is non-empty. The definition here, then, matches that given in Section 2.2, so we need make no further distinction.

5.1.6: Homology d-spheres have the nice property that they have no boundary, for if $\sum_{i=1}^{k} \alpha_i \sigma_i$ is a non-zero element of $H_d(\Sigma)$, then

$$0 = \partial\left(\sum_{i=1}^{k} \alpha_i \sigma_i\right) = \sum_{i=1}^{k} \alpha_i \partial(\sigma_i)$$

and so every (d-1)-face of each σ_i must be contained in at least one other, and hence exactly one other, σ_j $j \neq i$. Therefore $\Sigma' = cl(\bigcup_{i=1}^{k} \sigma_i)$ has no boundary. But then any d-simplex in $\Sigma\setminus\Sigma'$ can intersect Σ' in at most a (d-2)-simplex. Hence if τ is a maximal simplex in $\Sigma' \cap cl(\Sigma\setminus\Sigma')$, then

$$\ell k_{\Sigma'}\tau \cap \ell k_{cl(\Sigma\setminus\Sigma')}\tau = \{\emptyset\},$$

implying $\ell k_{\Sigma'}\tau$ and $\ell k_{cl(\Sigma\setminus\Sigma')}\tau$ are not connected by a path, so that $\ell k_{\Sigma}\tau = \ell k_{\Sigma'}\tau \cup \ell k_{cl(\Sigma\setminus\Sigma')}\tau$ has at least two path components, and $\tilde{H}_0(\ell k_{\Sigma}\tau) \neq 0$ ([26] Example 4.2.13). But the choice of τ means that $\dim(\ell k_{\Sigma}\tau) \geq 1$ and therefore $\ell k_{\Sigma}\tau$ cannot be a homology ball or sphere. Thus $\Sigma\setminus\Sigma' = \emptyset$, and so $\Sigma' = \Sigma$ has no boundary.

5.1.7: Homology d-balls and d-spheres belong to a more general class of complexes called Cohen-Macaulay complexes, that is, complexes Σ for which every simplex σ in Σ (including $\sigma = \emptyset$) has $H_m(\ell k_{\Sigma}\sigma) = 0$ except possibly at $m = d-|\sigma|$. These complexes will have added importance in the following sections.

5.2 Homological Properties of k-decomposable Complexes

We prove several homological characteristics of all k-decomposable complexes particularly homology balls and spheres, and present some interesting applications to shelling of spheres.

Theorem 5.2.1: k-decomposable complexes are Cohen-Macaulay.

<u>Proof</u>: Let Σ be a d-dimensional k-decomposable by Definition 2. If Σ is a d-simplex, then each face $\sigma \in \Sigma$ has $\ell k_\Sigma \sigma$ a simplex, and hence by Lemma 5.1.2 (ii) has 0 homology everywhere. Otherwise proceed by induction on $|\Sigma| > 2^{d+1}$, and let τ be a shedding simplex for Σ. We have $\Sigma\backslash\tau$ d-dimensional k-decomposable, and $\ell k_\Sigma \tau$ $(d-|\sigma|)$-dimensional k-decomposable, implying that $\bar{\tau}.\ell k_\Sigma \tau$ is d-dimensional k-decomposable and $(\bar{\tau}\backslash\tau).\ell k_\Sigma \tau$ is (d-1)-dimensional k-decomposable (Proposition 3.2.2). Further, $|\Sigma\backslash\tau| < |\Sigma|$, $|(\bar{\tau}\backslash\tau).\ell k_\Sigma \tau| < |\Sigma|$, and $|\bar{\tau}.\ell k_\Sigma \tau| < |\Sigma|$ (since $\Sigma\backslash\tau$ contains at least one d-simplex not in $\bar{\tau}.\ell k_\Sigma \tau$). Hence by induction $\tilde{H}_i(\Sigma\backslash\tau) = 0$, $\tilde{H}_i(\bar{\tau}.\ell k_\Sigma \tau) = 0$ except possibly at $i = d$, and $\tilde{H}_i((\bar{\tau}\backslash\tau).\ell k_\Sigma \tau) = 0$ except possibly at $i = d-1$. We can therefore set up the Mayer-Vietoris sequence as follows:

$$\cdots \to \tilde{H}_m((\bar{\tau}.\ell k_\Sigma \tau) \cap (\Sigma\backslash\tau)) \xrightarrow{\eta_m} \tilde{H}_m(\bar{\tau}.\ell k_\Sigma \tau) \oplus H_m(\Sigma\backslash\tau)$$

$$\xrightarrow{\xi_m} \tilde{H}_m((\bar{\tau}.\ell k_\Sigma \tau) \cup (\Sigma\backslash\tau)) \xrightarrow{\phi_m} \tilde{H}_{m-1}((\bar{\tau}.\ell k_\Sigma \tau) \cap (\Sigma\backslash\tau)) \to \cdots$$

which is the same as

$$\cdots \to \tilde{H}_m((\bar{\tau}\backslash\tau).\ell k_\Sigma \tau) \xrightarrow{\eta_m} \tilde{H}_m(\bar{\tau}.\ell k_\Sigma \tau) \oplus \tilde{H}_m(\Sigma\ \tau) \xrightarrow{\xi_m} \tilde{H}_m(\Sigma)$$

$$\xrightarrow{\phi_m} \tilde{H}_{m-1}((\bar{\tau}\backslash\tau).\ell k_\Sigma \tau) \to \cdots .$$

But for every $m \neq d$ we have

$$\cdots \to 0 \oplus 0 \xrightarrow{\xi_m} H_m(\Sigma) \xrightarrow{\phi_m} 0 \to \cdots$$

and since the sequence is exact,

$$\text{Ker } \phi_m = \text{Im } \xi_m = 0$$

so that ϕ_m is an injection into the zero space implying $\tilde{H}_m(\Sigma) = 0$, $m \neq d$, thus proving the theorem for $\sigma = \emptyset$. For $\sigma \neq \emptyset$ we simply note, by Proposition 3.2.1 that $\ell k_\Sigma \sigma$ is k-decomposable and that $|\ell k_\Sigma \sigma| < |\Sigma|$. Hence by induction $\tilde{H}_m(\ell k_\Sigma \sigma) = 0$ except possibly at $m = d-|\sigma|$. This completes the proof of the theorem.

<u>Theorem 5.2.2</u>: If Σ is a homology d-sphere and τ is a simplex in Σ, then $\Sigma\setminus\tau$ is a homology d-ball.

<u>Proof</u>: It is sufficient to prove only that $H_m(\Sigma\setminus\tau) = 0$ for all m, since for every simplex $\sigma \in \Sigma\setminus\tau$, if $\tau \notin \ell k_\Sigma \sigma$ then $\ell k_{\Sigma\setminus\tau}\sigma = \ell k_\Sigma \sigma$ and if $\tau \in \ell k_\Sigma \sigma$ then by Remark 5.1.6 $\ell k_{\Sigma\setminus\tau}\sigma = (\ell k_\Sigma \sigma)\setminus\tau$ is a homology $d-|\sigma|$ sphere with the simplex τ removed.

So now set up the Mayer-Vietoris sequence as in Theorem 5.2.1:

$$\cdots \to \tilde{H}_m((\bar{\tau}\setminus\tau).\ell k_\Sigma \tau) \xrightarrow{\eta_m} H_m(\bar{\tau}.\ell k_\Sigma \tau) \oplus \tilde{H}_m(\Sigma\setminus\tau) \xrightarrow{\xi_m} \tilde{H}_m(\Sigma)$$

$$\xrightarrow{\phi_m} \tilde{H}_{m-1}((\bar{\tau}\setminus\tau).\ell k_\Sigma \tau) \to \cdots .$$

We have by Lemma 5.1.2 and Remark 5.1.6

$$\tilde{H}_m((\bar{\tau}\setminus\tau).\ell k_\Sigma\tau) = \begin{cases} \mathbb{Z} & m = d-1 \\ 0 & m \neq d-1 \end{cases}$$

$$\tilde{H}_m(\bar{\tau}.\ell k_\Sigma\tau) = 0 \quad \text{all} \quad m$$

$$\tilde{H}_m(\Sigma) = \begin{cases} \mathbb{Z} & m = d \\ 0 & m \neq d \end{cases}$$

Hence for $m < d-1$ and $m > d$ the sequence is

$$0 \xrightarrow{\eta_m} H_m(\Sigma\setminus\tau) \xrightarrow{\xi_m} 0$$

implying $\tilde{H}_m(\Sigma\setminus\tau) = 0$. The remainder of the sequence is

$$0 \xrightarrow{\eta_d} \tilde{H}_d(\Sigma\setminus\tau) \xrightarrow{\xi_d} \mathbb{Z} \xrightarrow{\phi_d} \mathbb{Z} = \tilde{H}_{d-1}((\bar{\tau}\setminus\tau).\ell k_\Sigma\tau) \xrightarrow{\eta_{d-1}} \tilde{H}_{d-1}(\Sigma\setminus\tau) \to 0.$$

We concentrate first on the map η_{d-1} and prove that $\eta_{d-1} = 0$. Let $\sum_{i=1}^{k} a_i\sigma_i$ be a representation in $S_d(\Sigma)$ of the generator of $H_d(\Sigma)$. Then, as in Remark 5.1.6 $\sum_{i=1}^{k} a_i\sigma_i$ must include all the d-simplices in Σ. Consider $\gamma = \sum_{j=1}^{\ell} a_{i_j}\sigma_{i_j}$, where $\sigma_{i_1},\ldots,\sigma_{i_\ell}$ are all those σ_i in $\Sigma\setminus\tau$. Now $\partial_d(\gamma)$ has non-zero components exactly on all the $(d-1)$-simplices in $(\bar{\tau}\setminus\tau).\ell k_\Sigma\tau$, since cancellations occur exactly on the elements not in $(\Sigma\setminus\tau) \cap (\bar{\tau}.\ell k_\Sigma\tau)$. Further $\partial_{d-1} \circ \partial_d(\gamma) = 0$. Hence $\partial_d(\gamma)$ represents the unique generator of $H_{d-1}((\bar{\tau}\setminus\tau).\ell k_\Sigma\tau)$ (since $\operatorname{Im} \partial_d = 0$). But $\eta_{d-1}(\partial(\gamma)) = (0 \oplus)\partial(\gamma)$ which is 0 in

$\tilde{H}_{d-1}(\Sigma\backslash\tau)$ since it is in $\operatorname{Im}\partial_d$. Therefore, since η_{d-1} is onto, $\tilde{H}_{d-1}(\Sigma\backslash\tau) = 0$.

This means, however, that $\tilde{H}_d(\Sigma\backslash\tau) = 0$, since if ϕ_d is onto then $\phi_d(1) = \pm 1$ and hence $\operatorname{Ker}\phi_d = 0 = H_d(\Sigma\backslash\tau)$. Therefore $H_m(\Sigma\backslash\tau) = 0$ for all m, and hence $\Sigma\backslash\tau$ is a homology d-ball.

Theorem 5.2.3: Let Σ be a homology d-ball or d-sphere, τ a simplex in Σ. Then a necessary and sufficient condition for $\Sigma\backslash\tau$ to be Cohen-Macaulay is:

$$\sigma \in \partial\Sigma,\ \ \tau \in \ell k_\Sigma\sigma \Rightarrow \tau \in \partial\ell k_\Sigma\sigma. \tag{5.2.4}$$

Further, if (5.24) is satisfied then $\Sigma\backslash\tau$ is a homology d-ball.

Proof: Let σ be a simplex in $\Sigma\backslash\tau$. For $\tau \in \ell k_\Sigma\sigma$, we can set up the Mayer-Vietoris sequence as in Theorem 5.2.1, this time on $\ell k_\Sigma\sigma$ and τ instead of Σ and τ:

$$\cdots \to \tilde{H}_m((\bar{\tau}\backslash\tau).\ell k_{\ell k_\Sigma\sigma}\tau) \xrightarrow{\eta_m} \tilde{H}_m(\bar{\tau}.\ell k_{\ell k_\Sigma\sigma}\tau) \oplus \tilde{H}_m(\ell k_{\Sigma\backslash\tau}\sigma)$$
$$\xrightarrow{\xi_m} \tilde{H}_m(\ell k_\Sigma\sigma) \xrightarrow{\phi_m} \tilde{H}_{m-1}((\bar{\tau}\backslash\tau).\ell k_{\ell k_\Sigma\sigma}\tau) \to \cdots$$

noting by Lemma 2.3.3 that $(\ell k_\Sigma\sigma)\backslash\tau = \ell k_{\Sigma\backslash\tau}\sigma$. By the same arguments as Theorem 5.2.2 we have $\tilde{H}_m(\ell k_{\Sigma\backslash\tau}\sigma) = 0$ for $m \neq d-|\sigma|-1,\ d-|\sigma|$, and the remainder of the sequence is

$$0 \xrightarrow{\eta_{d-|\sigma|}} \tilde{H}_{d-|\tau|}(\ell k_{\Sigma\setminus\tau}\sigma) \xrightarrow{\xi_{d-|\sigma|}} \tilde{H}_{d-|\sigma|}(\ell k_{\Sigma}\sigma)$$

$$\xrightarrow{\phi_{d-|\sigma|}} \tilde{H}_{d-|\sigma|-1}((\overline{\tau}\setminus\tau).\ell k_{\ell k_{\Sigma}\sigma}\tau)$$

$$\xrightarrow{\eta_{d-|\sigma|-1}} \tilde{H}_{d-|\sigma|-1}(\ell k_{\Sigma\setminus\tau}\sigma) \xrightarrow{\xi_{d-|\sigma|-1}} 0.$$

First suppose that $\sigma \in \partial\Sigma$, $\tau \in \ell k_{\Sigma}\sigma$, and $\tau \notin \partial(\ell k_{\Sigma}\sigma)$. Then $H_{d-|\sigma|}(\ell k_{\Sigma}\sigma) = 0$ and $H_{d-|\sigma|-1}((\overline{\tau}\setminus\tau).\ell k_{\ell k_{\Sigma}\sigma}\tau) = \mathbb{Z}$ (Lemma 5.1.2). Hence we have

$$0 \xrightarrow{\phi_{d-|\sigma|}} \mathbb{Z} \xrightarrow{\eta_{d-|\sigma|-1}} \tilde{H}_{d-|\sigma|-1}(\ell k_{\Sigma\setminus\tau}\sigma) \xrightarrow{\xi_{d-|\sigma|-1}} 0$$

implying $\tilde{H}_{d-|\sigma|-1}(\ell k_{\Sigma\setminus\tau}\sigma) = \mathbb{Z}$, *so that* Σ *is not* Cohen-Macauley.

On the other hand, suppose (5.2.4) is satisfied. For any $\sigma \in \Sigma\setminus\tau$ we need to prove that $\ell k_{\Sigma\setminus\tau}\sigma$ has the homology of a ball or sphere, and the homology of a ball if $\sigma = \emptyset$.

Case 1 ($\tau \notin \ell k_{\Sigma}\sigma$): We have $\ell k_{\Sigma\setminus\tau}\sigma = \ell k_{\Sigma}\sigma$ and hence has the same homology.

Case 2 ($\sigma \notin \partial\Sigma$, $\tau \in \ell k_{\Sigma}\sigma$.): We have $\ell k_{\Sigma}\sigma$ is a homology sphere, and so by Theorem 5.2.2 $\ell k_{\Sigma\setminus\tau}\sigma$ is a homology ball.

Case 3 ($\sigma \in \partial\Sigma$, $\tau \in \ell k_{\Sigma}\sigma$.): We have by (5.2.4) $\tau \in \partial\ell k_{\Sigma}\sigma$, and so the remaining segment of the Mayer-Vietoris sequence is

$$0 \xrightarrow{\eta_{d-|\sigma|}} \tilde{H}_{d-|\sigma|}(\ell k_{\Sigma\setminus\tau}\sigma) \xrightarrow{\xi_{d-|\sigma|}} 0 \xrightarrow{\phi_{d-|\sigma|}} 0$$

$$\xrightarrow{\eta_{d-|\sigma|-1}} \tilde{H}_{d-|\sigma|-1}(\ell k_{\Sigma\setminus\tau}\sigma) \to 0,$$

implying $\tilde{H}_{d-|\sigma|}(\ell k_{\Sigma\setminus\tau}\sigma) = H_{d-|\sigma|-1}(\ell k_{\Sigma\setminus\tau}\sigma) = 0$ and so $\ell k_{\Sigma\setminus\tau}\sigma$ has 0 homology everywhere.

Cases 2 and 3 imply furthermore that $\Sigma\setminus\tau = \ell k_{\Sigma\setminus\tau}\emptyset$ is itself a homology ball, and hence the theorem is proved.

Notes: 1) Theorem 5.2.3 includes Theorems 5.2.2 as a special case, since if Σ is a d-sphere, then $\partial\Sigma = \emptyset$ and therefore (5.2.4) holds vacuously.

2) (5.2.4) must be tested for $\sigma = \emptyset$, which amounts to requiring that $\tau \in \partial\Sigma$ if Σ is a ball.

3) An equivalent statement of (5.2.4) as used in Theorem 4.1.3, is

$$\partial\Sigma \cap \ell k_{\Sigma}\tau = \partial\ell k_{\Sigma}\tau \tag{5.2.5}$$

since $\tau \in \ell k_{\Sigma}\sigma$ iff $\sigma \in \ell k_{\Sigma}\tau$, and $\tau \in \partial\ell k_{\Sigma}\sigma$ iff $\ell k_{\ell k_{\Sigma}\sigma}\tau = \ell k_{\ell k_{\Sigma}\tau}\sigma$ has the homology of a ball iff $\sigma \in \partial\ell k_{\Sigma}\tau$, so that (5.2.4) is equivalent to $\partial\Sigma \cap \ell k_{\Sigma}\tau \subseteq \partial\ell k_{\Sigma}\tau$. But $\partial\ell k_{\Sigma}\tau \subseteq \partial\Sigma$ always, hence (5.2.4) is equivalent to $\partial\Sigma \cap \ell k_{\Sigma}\tau = \partial\ell k_{\Sigma}\tau$.

We are now in a position to make a fourth definition of strong k-decomposability, namely:

Definition 4: A homology d-ball or d-sphere Σ is strongly k-decomposable if either Σ is a d-simplex or there is a simplex τ in Σ, $\dim\tau \leq k$, so that

1) (5.2.4) (or (5.2.5)) holds

2) $\Sigma \setminus \tau$ is strongly k-decomposable

3) $\ell k_{\Sigma} \tau$ is strongly k-decomposable.

Corollary 5.2.6: Definition 4 is equivalent to Definitions 1-3 for homology balls and spheres.

Proof: We prove Definition 4 equivalent to Definition 2.

($\Rightarrow$): We need only prove that $\Sigma \setminus \tau$, $\ell k_{\Sigma} \tau$ are homology d- and $(d-|\tau|)$-balls or spheres, since pure dimensionality follows. But this is clear from Theorem 5.2.3 and Remark 5.1.4.

($\Leftarrow$): Here we need only prove that if τ is a shedding simplex for (according to Definition 2) then (5.2.4) is satisfied. It is true vacuously for Σ a homology sphere, since then $\partial\Sigma = \emptyset$ by Remark 5.1.6. And if Σ is a homology ball, violation of (5.2.4) implies, by Theorems 5.2.1 and 5.2.3, that $\Sigma \setminus \tau$ is not Cohen-Macaulay, hence not k-decomposable.

5.3 Piecewise Linear Topology

Any set $S \subseteq \mathbb{R}^d$ will take the topology induced by the standard topology in $\mathbb{R}^d$. Two sets S and T in $\mathbb{R}^d$ are homeomorphic if there exists a continuous bijection $f: S \to T$ whose inverse is also continuous.

For Σ be a simplicial complex on E $E = \{v_1, \ldots, v_n\}$ a realization of Σ is the set

$$K_\Sigma = \bigcup_{\sigma = v_{i_1} \ldots v_{i_k} \in \Sigma} \operatorname{conv}\{x_{i_1}, \ldots, x_{i_k}\},$$

where x_i are points in R^m for some m which correspond to v_i, $i = 1,\ldots,n$, in such a way that the sets $\operatorname{ri}(\operatorname{conv}\{x_{i_1},\ldots,x_{i_k}\})$, $\sigma = v_{i_1} \ldots v_{i_k} \in \Sigma$, are disjoint. The sets $\operatorname{conv}\{x_{i_1},\ldots,x_{i_k}\}$ are called the geometric simplices (or just simplices) of K. Realizations exist for any complex ([26] Theorem 2.3.16). A subdivision of Σ is any simplicial complex Σ' with realization $K' = K_\Sigma$ such that every geometric simplex in K' is contained in some geometric simplex in K. One example of a subdivision is the stellar subdivision $st(a,X)[\Sigma]$, where the point x_a corresponding to a is the barycenter of the points corresponding to the vertices in X defined by

$$x_a = \frac{1}{|X|} \sum_{v_i \in X} x_i.$$

Finally, two simplicial complexes Σ_1, Σ_2 are called piecewise linear (PL) homeomorphic or combinatorially equivalent if there exists a single complex Σ' which is a subdivision of both Σ_1 and Σ_2. As a result, for realizations K_1 of Σ_1, K_2 of Σ_2, and respective realizations K_1' and K_2' of Σ' in K_1 and K_2, we can produce a "piecewise linear" homeomorphism $f\colon K_1 = K_1' \to K_2 = K_2'$ by mapping the vertices of K_1' into their respective counterparts in K_2' and extending linearly to the simplices of K_1' and K_2'. Of course, simplicial complexes are

always PL homeomorphic to themselves, and hence we can speak of homeomorphic or PL-homeomorphic complexes and subdivisions independent of a particular realization.

If Σ is PL homeomorphic to the d-simplex then Σ is called a PL or combinatorial d-ball, and if Σ is PL homeomorphic to the boundary of a (d+1)-simplex, then Σ is called a PL or combinatorial d-sphere.

Remarks:

5.3.1: Any two homeomorphic complexes have the same homology. Hence, since a geometric d-simplex (respectively the boundary of a d-simplex) is homeomorphic to the d-ball $B^d = \{x \in R^d | \sum_1^d x_i^2 \leq 1\}$ (respectively the d-sphere $S^d = \{x \in R^{d+1} | \sum_1^{d+1} x_i^2 = 1\})$ by placing the barycenter of the simplex at the origin and expanding radially from the origin, it follows (from [26] Examples 5.4.6, 5.3.2) that a PL d-ball or d-sphere is a homology d-ball or d-sphere.

5.3.2: The boundary of a PL d-ball is a PL (d-1)-sphere, since if Σ' is a subdivision of Σ which is also a subdivision of the d-simplex, then $\partial\Sigma'$ induces a subdivision on $\partial\Sigma$ which is also a subdivision of the boundary of the d-simplex.

5.3.3: J. Alexander [1] has shown that two complexes are PL-homeomorphic if and only if one can be obtained from the other by a series of stellar subdivisions and inverse stellar subdivisions, the latter being defined by

$$\Sigma' = st^{-1}(a,X)[\Sigma] \Longleftrightarrow \Sigma = st(a,X)[\Sigma'].$$

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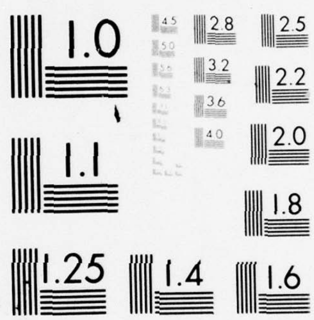

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Hence we can define a PL d-ball or d-sphere to be any complex which can be obtained from a combinatorial d-simplex or boundary of a combinatorial (d+1)-simplex, by a series of stellar subdivisions and inverse stellar subdivisions. We can then dispense with realizations altogether, and are left with a purely combinatorial definition of PL balls and spheres (though it by no means provides a finite algorithm for determining whether complexes are PL balls or spheres).

5.3.4: For $d \leq 3$ all d-balls and d-spheres (complexes whose realizations are homeomorphic to B^d or S^d) are PL [28]. For $d \geq 5$ there exist d-spheres and d-balls which are not PL [13,14]. For $d = 4$ the question is still open.

The following lemmas and theorems in PL theory will be needed to prove the results in the following section; and are stated without proof. They are referenced to the corresponding theorems in [19].

Theorem 5.3.5: If Σ_1, Σ_2 are two PL-homeomorphic complexes, then there is a complex Σ_3 which is a subdivision of Σ_1 and a stellar subdivision of Σ_2. (Corollary 1.6)

Lemma 5.3.6: If Σ is a PL d-ball or d-sphere and σ is a simplex in Σ, then $\ell k_\Sigma \sigma$ is a PL $(d-|\sigma|)$-ball or $(d-|\sigma|)$-sphere. (Corollary 1.16)

Lemma 5.3.7: If Σ_1 is a PL p-ball, and Σ_2 is a PL q-ball or q-sphere, then $\Sigma_1 \cdot \Sigma_2$ is a PL (p+q+1)-ball. If Σ_1 is a PL p-sphere and Σ_2 is a PL q-sphere, then $\Sigma_1 \cdot \Sigma_2$ is a PL (p+q+1)-sphere. (Lemma 1.1.3)

Theorem 5.3.8: If Σ_1 is a PL d-ball contained (as a complex) in some PL d-sphere Σ_2, then $cl(\Sigma_2 \setminus \Sigma_1)$ is a PL d-ball. (Theorem 1.26)

Theorem 5.3.9: If Σ_1 and Σ_2 are two PL d-balls whose intersection is a PL (d-1)-ball contained in the boundary of Σ_1 and Σ_2, then $\Sigma_1 \cup \Sigma_2$ is a PL d-ball. (Corollary 1.28)

5.4 PL Properties of k-Decomposable Complexes

Theorem 5.4.1: Every PL ball or sphere has a subdivision which is vertex decomposable.

Proof: Let Σ be a PL d-ball (d-sphere, respectively). By Theorem 5.3.5 there exists a subdivision Σ' of Σ which is also a stellar subdivision of the d-simplex (boundary of the (d+1)-simplex, respectively). Now the d-simplex and boundary of the (d+1)-simplex are both vertex decomposable, and by Proposition 3.2.4 vertex decomposition is preserved under stellar subdivisions. Hence Σ' is a subdivision of Σ which is vertex decomposable.

Corollary 5.4.2: Every PL ball or sphere has a subdivision which satisfies the Hirsch Conjecture.

An interesting general characteristic of vertex decomposability comes directly from Remark 5.3.3 and Proposition 3.2.4 (and can be extended to k-decomposability by use of Appendix 1).

Proposition 5.4.3: Let Σ be a vertex decomposable complex, and $\mathcal{C}(\Sigma)$ the class of all complexes PL-homeomorphic to Σ. Then $\mathcal{C}(\Sigma)$ is a vertex decomposable class iff vertex decomposability is preserved under inverse stellar subdivision for elements in the class.

We can prove results similar to Theorems 5.2.1, 5.2.2, 5.2.3, and 5.2.6 about removal of simplices from PL balls and spheres. We first need a lemma.

Lemma 5.4.4: If Σ is a PL d-ball and v a vertex not in Σ, then $\Sigma \cup v.\partial\Sigma$ is a PL d-sphere.

Proof: Let Σ' be a subdivision of Σ which is also a subdivision of the d-simplex Δ^d. Then it is clear that $\Sigma' \cup \bar{v}.\partial\Sigma'$ is a subdivision of $\Sigma \cup \bar{v}.\partial\Sigma$ which is also a subdivision of $\Delta^d \cup \bar{v}.\partial\Delta^d = \partial\Delta^{d+1}$. Hence $\Sigma \cup \bar{v}.\partial\Sigma$ is a PL d-sphere.

The following result is known ([10] Proposition 1.2) but we give it here because of its relevance and simplicity of proof. Define a d-dimensional complex to be a pseudo-manifold if every (d-1)-simplex in Σ is contained in at most two d-simplices of Σ. (This definition differs slightly from that in, say, [22]).

Theorem 5.4.5: A k-decomposable d-dimensional pseudo-manifold is a PL d-ball or d-sphere.

Proof: Let Σ be a d-dimensional k-decomposable pseudo-manifold. We proceed by induction on $p(\Sigma)$ = the number of d-simplices in Σ. If $p(\Sigma) = 1$ then Σ is a d-simplex, and hence a PL d-ball. If $p(\Sigma) > 1$, then Σ has a shedding simplex Σ by Definition 2. We have the decomposition $\Sigma = (\Sigma \setminus \tau) \cup (\bar{\tau}.\ell k_{\Sigma}\tau)$, where $\Sigma \setminus \tau$, $\bar{\tau}.\ell k_{\Sigma}\tau$ are k-decomposable pseudo-manifolds, have fewer d-simplices than Σ, and so by induction are PL d-balls or d-spheres. Further, since Σ is

a d-dimensional pseudo-manifold, the complex

$$(\Sigma\backslash\tau) \cap (\bar{\tau}.\ell k_{\Sigma}\tau) = (\bar{\tau}\backslash\tau).\ell k_{\Sigma}\tau \neq \emptyset$$

is in the boundary of both $\Sigma\backslash\tau$ and $\bar{\tau}.\ell k_{\Sigma}\tau$. Hence $\Sigma\backslash\tau$ and $\tau.\ell k_{\Sigma}\tau$ are in fact d-balls, and $\ell k_{\Sigma}\tau = \ell k_{\bar{\tau}.\ell k_{\Sigma}\tau}\tau$ is consequently either a PL $(d-|\tau|)$-ball or $(d-|\tau|)$-sphere by Lemma 5.3.6.

Case 1 ($\ell k_{\Sigma}\tau$ is a PL $(d-|\tau|)$-sphere): We have $(\bar{\tau}\backslash\tau).\ell k_{\Sigma}\tau$ is a PL (d-1)-sphere (Lemma 5.3.7) which is contained in the PL (d-1)-sphere $\partial(\Sigma\backslash\tau)$ (Remark 5.3.2). So by discussion similar to that of Remark 5.1.6, $\partial(\Sigma\backslash\tau) = (\bar{\tau}\backslash\tau).\ell k_{\Sigma}\tau$. Therefore the stellar subdivision of Σ

$$\begin{aligned} st(v,\tau)[\Sigma] &= \Sigma\backslash\tau \cup v.(\bar{\tau}\backslash\tau).\ell k_{\Sigma}\sigma \\ &= \Sigma\backslash\tau \cup v.\partial(\Sigma\backslash\tau) \end{aligned}$$

is a PL d-sphere (Lemma 5.4.4), hence so is Σ.

Case 2 ($\ell k_{\Sigma}\tau$ is a PL $(d-|\tau|)$-ball): We have $(\bar{\tau}\backslash\tau).\ell k_{\Sigma}\tau$ is a PL (d-1)-ball (Lemma 5.3.7). Hence by Theorem 5.3.9, $\Sigma = (\Sigma\backslash\tau) \cup ((\bar{\tau}\backslash\tau).\ell k_{\Sigma}\tau)$ is a PL d-ball. This completes the proof.

As a result of the proof, we see that a characterization of whether such a complex Σ is a ball or sphere (providing Σ is not a simplex) is simply whether some (equivalently any) shedding simplex τ has $\ell k_{\Sigma}\tau$ a ball or sphere.

<u>Corollary 5.4.6</u>: The dual complex of a polyhedron is a PL ball or sphere.

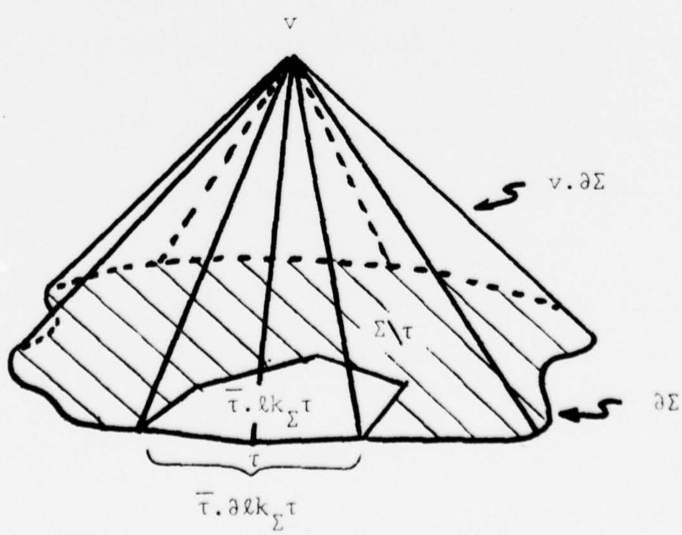
Proof: The dual complex of a polyhedron is shellable = d-decomposable (Section 2.5), and is a pseudo-manifold since every edge of a polyhedron contains at most two vertices.

We now give the result corresponding to Theorems 5.2.2 and 5.2.3 on partial decomposition of PL balls and spheres.

Theorem 5.4.7: If Σ is a PL d-ball or d-sphere, $\tau \in \Sigma$, then $\Sigma\setminus\tau$ is a PL d-ball iff (5.2.4) (or (5.2.5)) holds.

Proof: The necessity holds from Theorem 5.2.3 (since Σ is also a homology ball or sphere). Further, if Σ is a PL d-sphere, then $\bar{\tau}.\ell k_\Sigma \tau$ is a PL d-ball (Lemmas 5.3.6 and 5.3.7) and so by Theorem 5.3.9 $\Sigma\setminus\tau = cl(\Sigma\setminus\bar{\tau}.\ell k_\Sigma\tau)$ is a PL d-ball. Hence sufficiency holds for PL spheres.

Now suppose that Σ is a PL d-ball and τ satisfies (5.2.4). Then $\Sigma \cup \bar{v}.\partial\Sigma$ is a PL d-sphere (Lemma 5.4.4). Consider the subcomplex $\bar{v}.\partial\Sigma \cup \bar{\tau}.\ell k_\Sigma \tau$.



We have

$$\bar{v}.\partial\Sigma \cap \bar{\tau}.\ell k_{\Sigma}\tau = \partial\Sigma \cap \bar{\tau}.\ell k_{\Sigma}\tau$$

$$= \bar{\tau}.\partial\ell k_{\Sigma}\tau,$$

since, for $\mu.\sigma \in \bar{\tau}.\ell k_{\Sigma}\tau$, $\mu.\sigma \in \partial\Sigma$ implies $\sigma \in \partial\Sigma$, which in turn implies $\sigma \in \partial\ell k_{\Sigma}\tau$ (since (5.2.4) holds for Σ), and conversely $\mu.\sigma \notin \partial\Sigma$ implies $\tau.\sigma(\supseteq \mu.\sigma) \notin \partial\Sigma$, which means that $\ell k_{\ell k_{\Sigma}\tau}\sigma = \ell k_{\Sigma}(\tau \cup \sigma)$ has the homology of a sphere, and so $\sigma \notin \partial\ell k_{\Sigma}\tau$. Now by Lemma 5.3.6, Lemma 5.3.7, and Remark 5.3.2 $\bar{v}.\partial\Sigma$ and $\bar{\tau}.\ell k_{\Sigma}\tau$ are PL d-balls, and $\bar{\tau}.\partial\ell k_{\Sigma}\tau$ is a PL (d-1)-ball contained in the boundary of each (since Σ is a manifold). So (Theorem 5.3.9) $\bar{v}.\partial\Sigma \cup \bar{\tau}.\ell k_{\Sigma}\tau$ is a PL d-ball.

We are now in a position to apply Theorem 5.3.8. Removing $\bar{v}.\partial\Sigma \cup \bar{\tau}.\ell k_{\Sigma}\tau$ from $\Sigma \cup \bar{v}.\partial\Sigma$, we get

$$\mathrm{cl}((\Sigma \cup \bar{v}.\partial\Sigma)\setminus(\bar{v}.\partial\Sigma \cup \bar{\tau}.\ell k_{\Sigma}\tau)) = \Sigma\setminus\tau$$

is a PL d-ball, and this completes the proof of the theorem.

To end Chapter 5, we give an interesting application of Theorem 5.4.7 (or just as easily Theorem 5.3.4) to shelling of spheres.

<u>Theorem 5.4.8</u>: Let Σ be a d-sphere (topological, homology, or PL) and $\sigma_1,\ldots,\sigma_k$ a shelling of the d-simplices of Σ, then $\sigma_k,\ldots,\sigma_1$ is also a shelling of Σ.

Proof: Recall $\sigma_1,\ldots,\sigma_k$ is a shelling of Σ if and only if $\bar{\sigma}_i \cap (\bigcup_{j=1}^{i-1} \bar{\sigma}_j)$ is a pure (d-1)-dimensional complex, $i = 2,\ldots,k$. We prove that $\bar{\sigma}_i \cap (\bigcup_{j=i+1}^{k} \bar{\sigma}_j)$ is pure (d-1)-dimensional, $i = 1,\ldots,k-1$. By Theorem 3.3, we can consider the shelling a d-decomposition which removes one d-simplex at a time (starting with σ_k). Hence by Theorem 5.2.3 and Theorem 5.4.7, the complex $\Sigma_i = (\bigcup_{j=1}^{i} \bar{\sigma}_j)$ is a PL d-ball for $i \leq k-1$ with shedding simplex τ_i so that $\Sigma_i \backslash \tau_i = cl(\Sigma_i \backslash \{\bar{\sigma}_i\}) = (\bigcup_{j=1}^{i-1} \bar{\sigma}_j) = \Sigma_{i-1}$. Now since Σ is a sphere, $\Sigma_i \cap (\bigcup_{j=i+1}^{k} \bar{\sigma}_j) = \partial\Sigma_i$. But since τ_i is a shedding simplex for Σ_i, Condition 5.2.5 holds. Hence exactly as in the proof of Theorem 5.4.7 we have

$$\begin{aligned}\partial\Sigma_i \cap \bar{\sigma}_i &= \partial\Sigma_i \cap \bar{\tau}_i . \ell k_{\Sigma_i} \tau_i \\ &= \bar{\tau}_i . \partial\ell k_{\Sigma_i} \tau_i\end{aligned}$$

which is a PL (d-1)-ball and consequently pure (d-1)-dimensional. So

$$\bar{\sigma}_i \cap (\bigcup_{j=i+1}^{k} \bar{\sigma}_j) = \bar{\sigma}_i \cap \partial\Sigma_i$$

is a pure (d-1)-dimensional complex, thus completing the proof of the theorem.

Corollary 5.4.9: If Σ_1 and Σ_2 are two shellable d-balls whose union is a d-sphere and whose intersection is a shellable (d-1)-sphere, then $\Sigma_1 \cup \Sigma_2$ is shellable.

<u>Proof</u>: We have, by Lemma 5.4.4 that $\Sigma_1 \cup \bar{v}.\partial\Sigma_1$ is a d-sphere for v not a vertex in Σ_1. Now $\Sigma_1 \cup \bar{v}.\partial\Sigma_1$ is shellable by the shelling order of Σ_1 followed by $v.\tau_1,\ldots,v.\tau_k$, where $\tau_1,\ldots,\tau_k$ is a shelling of $\partial\Sigma_1 = \partial\Sigma_2 = \Sigma_1 \cap \Sigma_2$. Then by Theorem 5.4.8, the reverse ordering is also a shelling order. But by substituting for $v.\tau_1,\ldots,v.\tau_k$ the shelling order of Σ_2, we produce a shelling of $\Sigma_1 \cup \Sigma_2$.

APPENDIX 1

k-DECOMPOSITION AND STELLAR SUBDIVISIONS

Theorem A1: (Weak) k-decomposability is preserved under stellar subdivisions.

Proof: Let Σ be k-decomposable, $X \neq \emptyset$ a simplex in Σ, and a the additional vertex. We have $|\Sigma| \geq 2$, and for $\Sigma = \bar{\sigma}$ a simplex,

$$\begin{aligned} st(a,X)[\Sigma] &= (\bar{\sigma}\backslash X) \cup a.\partial X.\ell k_{\bar{\sigma}} X \\ &= (\overline{\sigma\backslash X}) \cup a.\partial X.(\overline{\sigma\backslash X}) \\ &= \bar{a}.\partial X.(\overline{\sigma\backslash X}) \end{aligned}$$

which is (weakly) k-decomposable since each component is (weakly) k-decomposable. Now proceed by induction on $|\Sigma| > 2$, Σ not a simplex, and let τ be a shedding simplex for Σ, $\dim \tau \leq k$, so that $\Sigma\backslash\tau$ is (weakly) k-decomposable (and $\ell k_{\Sigma}\tau$ is k-decomposable).

We prove the following:

A. $\tau \cup X \notin \Sigma$ implies 1) $\ell k_{st(a,X)[\Sigma]}\tau = \ell k_{\Sigma}\tau$, and
2) $st(a,X)[\Sigma]\backslash\tau = st(a,X)[\Sigma\backslash\tau]$.

B. $\tau \in \ell k_{\Sigma} X$ implies 1) $\ell k_{st(a,X)[\Sigma]}\tau = st(a,X)[\ell k_{\Sigma}\tau]$, and
2) $st(a,X)[\Sigma]\backslash\tau = st(a,X)[\Sigma\backslash\tau]$.

C. $\tau \cup X \in \Sigma$, $X \not\subseteq \tau$, $X \cap \tau \neq \emptyset$ imply 1) $\ell k_{st(a,X)[\Sigma]}\tau = st(a,X\backslash\tau)[\ell k_\Sigma \tau]$,

2) $\ell k_{st(a,X)[\Sigma]\backslash\tau}\, a.(\tau\backslash X) = (\overline{X\backslash\tau}).\ell k_\Sigma(X \cup \tau)$, and

3) $st(a,X)[\Sigma]\backslash\tau\backslash a.(\tau\backslash X) = st(a,X)[\Sigma\backslash\tau]$ (where we define

$st(a,X)[\Sigma\backslash\tau] \equiv \Sigma\backslash\tau$ for $X \notin \Sigma\backslash\tau$).

D. $X \subseteq \tau$ implies 1) $\ell k_{st(a,X)[\Sigma]}\, a.(\tau\backslash X) = \partial X.\ell k_\Sigma \tau$ and

2) $st(a,X)[\Sigma]\backslash a.(\tau\backslash X) = st(a,X)[\Sigma\backslash\tau]$.

The right hand sides are all (weakly) k-decomposable by induction on the number of simplices and applications of Propositions 3.2.1 and 3.2.2 (noting in C and D that $\dim(a.(\tau\backslash X)) \leq k$), and since A,B,C,D, cover all cases for τ, the theorem follows.

A. 1) $\ell k_{st(a,X)[\Sigma]}\tau = \ell k_{\Sigma\backslash X}\tau$ (Lemma 2.3.1, since $\tau \notin \bar{a}.\partial X.\ell k_\Sigma X$)

$= \ell k_\Sigma \tau$ (since $\tau \cup X \notin \Sigma$)

2) $st(a,X)[\Sigma]\backslash\tau = [(\Sigma\backslash X)\backslash\tau] \cup a.\partial X.\ell k_\Sigma X$ (Lemma 2.3.1, since $\tau \notin a.\partial X.\ell k_\Sigma X$)

$= [(\Sigma\backslash\tau)\backslash X] \cup a.\partial X.\ell k_{\Sigma\backslash\tau} X$ (Lemma 2.3.3 and $X \cup \tau \notin \Sigma$)

$= st(a,X)[\Sigma\backslash\tau]$

B. 1) $\ell k_{st(a,X)[\Sigma]}\tau = \ell k_{\Sigma\backslash X}\tau \cup \ell k_{\bar{a}.\partial X.\ell k_\Sigma X}\tau$ (since $\tau \in (\Sigma\backslash X) \cap (\bar{a}.\partial X.\ell k_\Sigma X)$)

$= (\ell k_\Sigma \tau)\backslash X \cup \ell k_{\bar{a}}\emptyset.\ell k_{\partial X}\emptyset.\ell k_{\ell k_\Sigma X}\tau$ (Lemma 2.3.2)

$= (\ell k_\Sigma \tau)\backslash X \cup \bar{a}.\partial X.\ell k_\Sigma(X \cup \tau)$ (Lemma 2.3.4)

$= (\ell k_{\Sigma}\tau)\backslash X \cup \bar{a}.\partial X.\ell k_{\ell k_{\Sigma}\tau} X$ (*Lemma 2.3.4*)

$= st(a,X)[\ell k_{\Sigma}\tau]$.

2) $st(a,X)[\Sigma]\backslash\tau = [(\Sigma\backslash X)\backslash\tau] \cup [(\bar{a}.\partial X.\ell k_{\Sigma} X)\backslash\tau]$ (Lemma 2.3.1, since $\tau \in (\Sigma\backslash X) \cap (\bar{a}.\partial X\ell k_{\Sigma} X))$

$= (\Sigma\backslash\tau)\backslash X \cup \bar{a}.\ell X.[(\ell k_{\Sigma} X)\backslash\tau]$ (Lemma 2.3.3 and Lemmma 2.3.2, since $\tau \notin \bar{a}$, $\tau \notin \partial X$)

$= (\Sigma\backslash\tau)\backslash X \cup \bar{a}.\partial X.\ell k_{\Sigma\backslash\tau} X$ (Lemma 2.3.3)

$= st(a,X)[\Sigma\backslash\tau]$.

C. 1) $\ell k_{st(a,X)[\Sigma]}\tau = \ell k_{\Sigma\backslash X}\tau \cup \ell k_{\bar{a}.\partial X.\ell k_{\Sigma} X}\tau$ (Lemma 2.3.1, since $\tau \notin (\Sigma\backslash X) \cap (\bar{a}.\partial X.\ell k_{\Sigma} X))$

$= \ell k_{\Sigma\backslash(X\backslash\tau)}\tau \cup [\ell k_{\bar{a}}(\tau \cap a)].[\ell k_{\partial X}(\tau \cap X)].[\ell k_{\ell k_{\Sigma} X}(\tau\backslash X)]$

(Lemma 2.3.5 and Lemma 2.7.2)

$= [(\ell k_{\Sigma}\tau)\backslash(X\backslash\tau)] \cup a.\partial(X\backslash(\tau \cap X)).\ell k_{\Sigma}\tau \cup X$

(Lemma 2.3.3 and Lemma 4)

$= [(\ell k_{\Sigma}\tau)\backslash(X\backslash\tau)] \cup a.\partial(X\backslash\tau).\ell k_{\ell k_{\Sigma}\tau} X\backslash\tau$

$= st(a,X\backslash\tau)[\ell k_{\Sigma}\tau]$.

$$2)\quad st(a,X)[\Sigma]\backslash\tau = [(\Sigma\backslash X)\backslash\tau] \cup [(\bar{a}.\partial X.\ell k_\Sigma X)\backslash\tau] \quad \text{(Lemma 2.3.1i)}$$

$$= [(\Sigma\backslash X)\backslash\tau] \cup \bar{a}.[\partial X\backslash(\tau \cap X)].\ell k_\Sigma X$$

$$\cup\ \bar{a}.\partial X.[(\ell k_\Sigma X)\backslash(\tau\backslash X)]$$

So $\ell k_{st(a,X)[\Sigma]}a.(\tau\backslash X) = \ell k_{\bar{a}.[\partial X\backslash(\tau \cap X)].\ell k_\Sigma X}a.(\tau\backslash X)$ (Lemma 2.3.1,

$a.(\tau\backslash X) \notin (\Sigma\backslash X)\backslash\tau$ and

$a.(\tau\backslash X) \notin \bar{a}.\partial X.[(\ell k_\Sigma X)\backslash(\tau\backslash X)]$)

$$= (\ell k_{\bar{a}}a).\ell k_{\partial X\backslash(\tau \cap X)}\emptyset.\ell k_{\ell k_\Sigma X}(\tau\backslash X)$$

$$= \{\emptyset\}.\partial X\backslash(\tau \cap X).\ell k_\Sigma \tau \cup X$$

$$= (\overline{X\backslash\tau}).\ell k_\Sigma \tau \cup X$$

$$3)\quad st(a,X)[\Sigma]\backslash\tau = [(\Sigma\backslash X)\backslash\tau] \cup \bar{a}.(\overline{X\backslash\tau}).\ell k_\Sigma X \cup \bar{a}.\partial X.[(\ell k_\Sigma X)\backslash(\tau\backslash X)]$$

(from 2)

$$st(a,X)[\Sigma]\backslash\tau\backslash a.(\tau\backslash X) = [(\Sigma\backslash X)\backslash\tau] \cup [(\bar{a}.(\overline{X\backslash\tau}).\ell k_\Sigma X)\backslash(a.(\tau\backslash X))]$$

$$\cup\ \bar{a}.\partial X.[(\ell k_\Sigma X)\backslash(\tau\backslash X)]$$

(Lemma 2.3.1, since $a.(\tau\backslash X) \notin (\Sigma\backslash X)\backslash\tau$

$a.(\tau\backslash X) \notin a.\partial X.[(\ell k_\Sigma X).(\tau\backslash X)]$)

$$= [(\Sigma\backslash X)\backslash\tau] \cup [\bar{a}.(\overline{X\backslash\tau}).[(\ell k_\Sigma X)\backslash(\tau\backslash X)]$$

$$\cup\ (\overline{X\backslash\tau}).\ell k_\Sigma X] \cup \bar{a}.\partial X.[(\ell k_\Sigma X)\backslash(\tau\backslash X)]$$

(Lemma 2.3.2, since $a.(\tau\backslash X) \cap X\backslash\tau = \emptyset$

and $(\bar{a}\backslash a).(\overline{X\backslash\tau}).\ell k_\Sigma X = (\overline{X\backslash\tau}).\ell k_\Sigma X$)

$$= [(\Sigma\backslash X)\backslash\tau] \cup \bar{a}.\partial X.[(\ell k_\Sigma X)\backslash(\tau\backslash X)]$$

(since $(\overline{X\backslash\tau}).\ell k_\Sigma X \subseteq (\Sigma\backslash X)\backslash\tau$,

$\bar{a}.(\overline{X\backslash\tau}) \subseteq \bar{a}.\partial X)$

$$= \begin{cases} (\Sigma\backslash\tau)\backslash X \cup \bar{a}.\partial X.\ell k_{\Sigma\backslash\tau} X & \tau \not\subseteq X \\ \quad \text{(by Lemma 2.3.3, Lemma 2.3.5)} & \\ (\Sigma\backslash\sigma)\backslash X = \Sigma\backslash\tau & \tau \subseteq X \\ \quad \text{(since } \ell k_\Sigma X\backslash\{\emptyset\} = \emptyset) & \end{cases}$$

$$= st(a,X)[\Sigma\backslash\tau].$$

D. 1) $\ell k_{st(a,X)[\Sigma]} a.(\tau\backslash X) = \ell k_{\bar{a}.\partial X.\ell k_\Sigma X} a.(\tau\backslash X)$ (Lemma 2.3.2, since $a.(\tau\backslash X) \not\in \Sigma\backslash X)$

$$= (\ell k_{\bar{a}} a).\ell k_{\partial X}\emptyset.\ell k_{\ell k_\Sigma X}(\tau\backslash X) \quad \text{(Lemma 2.3.2)}$$

$$= \{\emptyset\}.\partial X.\ell k_\Sigma X \cup \tau$$

$$= \partial X.\ell k_\Sigma \tau$$

2) $st(a,X)[\Sigma]\backslash a.(\tau\backslash X) = (\Sigma\backslash X) \cup [(\bar{a}.\partial X.\ell k_\Sigma X)\backslash a.(\tau\backslash X)]$

(Lemma 2.3.1, since $a.(\tau\backslash X) \not\in \Sigma\backslash X)$

$$= (\Sigma\backslash X) \cup \bar{a}.\partial X.[\ell k_\Sigma X\backslash(\sigma\backslash X)] \cup \partial X.\ell k_\Sigma X$$

(Lemma 2.3.2, since $a.(\tau\backslash X) \cap X = \emptyset$

$(\bar{a}\backslash a).\partial X.\ell k_\Sigma X = \partial X.\ell k_\Sigma X)$

$= (\Sigma \backslash X) \cup \bar{a}.\partial X.\ell k_{\Sigma \backslash (\tau \backslash X)}X$ (Lemma 2.3.3 and $\partial X.\ell k_{\Sigma}X \subseteq \Sigma \backslash X$)

$= (\Sigma \backslash \tau) \backslash X \cup \bar{a}.\partial X.\ell k_{\Sigma \backslash \tau}X$ ($X \subseteq \tau$ and Lemma 2.3.5)

$= st(a,X)[\Sigma \backslash \tau]$

This completes the proof.

APPENDIX 2

DIAMETERS OF GENERAL SIMPLICIAL COMPLEXES

Let $C(n,d)$ be the class of (simplicially) path connected d-dimensional complexes on n vertices. We present upper and lower bounds for the maximum of the diameters of elements in $C(n,d)$.

Proposition A2.1: The diameter of any element of $C(n,d)$ is at most $\binom{n}{d}/d$.

Proof: Let Σ be an element of $C(n,d)$, and Γ: $\sigma_0,\sigma_1,\ldots,\sigma_p$ a shortest simplicial path between the simplices σ_0 and σ_p. Notice that the only d-simplices in Γ which intersect in a (d-1)-face must be adjacent in the ordering, otherwise Γ would not be a shortest path from σ_0 to σ_p. So the number of (d-1)-simplices in such a complex is $d+1$ times the number of d-simplices in Γ minus the number of (d-1)-simplices which are in the intersection of two d-simplices, or $(d+1)p-p = dp$. But the total number of (d-1)-simplices in Γ is at most $\binom{n}{d}$, and therefore

$$p \leq \binom{n}{d}/d.$$

Note: The ratio of $\binom{n}{d}/d$ to $\binom{n}{d+1}$, the number of d-simplices in Γ, is $\frac{d+1}{d(n-d)}$, so this is a slight improvement over a straight d-simplex count.

Proposition A2.2: There exists a complex $\Sigma \in C(n,d-1)$ whose diameter is $\lfloor n/d \rfloor^{\lfloor d/2 \rfloor}$ ($\lfloor\ \rfloor$ = least integer).

Proof: *Let* V *be the set of* n *vertices.* Divide V into $\lfloor d/2 \rfloor$ groups $S_i = \{v_1^i, \ldots, v_{p_i}^i\}$, where $p_i = \lfloor 2n/d \rfloor$, $i < \lfloor d/2 \rfloor$, $p_{\lfloor d/2 \rfloor} = n - (\lfloor d/2 \rfloor - 1)\lfloor 2n/d \rfloor$. Note that if 2 does not divide d, then $p_{\lfloor d/2 \rfloor} > \lfloor 2n/d \rfloor$. Start with $\Gamma_0 = \{\emptyset\}$, and suppose by induction that Γ_k is a shortest simplicial path of dimension $2k$ between the $2k$-simplices Δ_0, Δ_1 on the vertex set $\bigcup_{i=1}^{k} S_i$ with $\lfloor n/d \rfloor^k$ $2k$-simplices, for $0 \leq i < \lfloor d/2 \rfloor$. Denote Γ_k^{-1} the reverse path from Δ_1 to Δ_0. Define

$$\Gamma_{k+1} = \Gamma_k . v_1^{k+1} v_2^{k+1} \cup \Delta_1 . v_2^{k+1} v_3^{k+1} \cup \Gamma_k^{-1} . v_3^{k+1} v_4^{k+1} \cup \Delta_0 . v_4^{k+1} v_5^{k+1}$$
$$\cup\ \Gamma_k . v_5^{k+1} v_6^{k+1} \cup \ldots$$

if 2 divides d or $i < \lfloor d/2 \rfloor$ and

$$\Gamma_{k+1} = \Gamma_k . v_1^{k+1} v_2^{k+1} v_3^{k+1} \cup \Delta_1 . v_2^{k+1} v_3^{k+1} v_4^{k+1} \cup \Gamma_k^{-1} . v_3^{k+1} v_4^{k+1} v_5^{k+1}$$
$$\cup\ \Delta_0 . v_4^{k+1} v_5^{k+1} v_6^{k+1} \cup \Gamma_k . v_5^{k+1} v_6^{k+1} v_7^{k+1} \cup \ldots$$

if 2 does not divide d and $i = \lfloor d/2 \rfloor$. Then the d-simplex pairs which intersect at a (d-1)-simplex are precisely those adjacent in the ordering given, hence Γ_{k+1} is a shortest simplicial path between the simplices $\Delta_0 . v_1 v_2 (v_3)$ and $\Delta_\delta . (v_{p_i-2}^{k+1}) v_{p_i-1}^{k+1} v_{p_i}^{k+1}$ where

$$\delta = \begin{cases} 0 & p_i \ (p_i - 1 \text{ respectively}) \equiv 0 \text{ or } 1 \mod 4 \\ 1 & p_i \ (p_i - 1 \text{ respectively}) \equiv 2 \text{ or } 3 \mod 4 \end{cases}$$

The number of 2k-simplices in Γ_{k+1} is at least

$$\frac{\lfloor 2n/d \rfloor}{2}(\lfloor n/d \rfloor^k+1) = \lfloor n/d \rfloor(\lfloor n/d \rfloor^k+1)$$

$$= \lfloor n/d \rfloor^{k+1} + \lfloor n/d \rfloor \geq \lfloor n/d \rfloor^{k+1}+1$$

and so for $\Sigma = \mathrm{cl}(\Gamma_{\lfloor d/2 \rfloor})$

$$\mathrm{diam}\ \Sigma \geq \lfloor n/d \rfloor^{\lfloor d/2 \rfloor}.$$

An example of such a complex for $n = 14$, $d = 7$ is given in Table 6. Its diameter is 15, which is greater than $\lfloor 14/7 \rfloor^{\lfloor 7/2 \rfloor} = 8$. A more careful analysis of the proof of Proposition A2.2 shows in fact that diam Σ is at least $\lfloor n/d \rfloor^{\lfloor d/2 \rfloor} + \lfloor n/d \rfloor^{\lfloor d/2 \rfloor - 1} \ldots + \lfloor n/d \rfloor$.

S_1				S_2				S_3					
v_1^1	v_2^1	v_3^1	v_4^1	v_1^2	v_2^2	v_3^2	v_4^2	v_1^3	v_2^3	v_3^3	v_4^3	v_5^4	v_6^3
1	1			1	1			1	1	1			
	1	1		1	1			1	1	1			
		1	1	1	1			1	1	1			
		1	1		1	1		1	1	1			
		1	1			1	1	1	1	1			
	1	1				1	1	1	1	1			
1	1					1	1	1	1	1			
1	1					1	1		1	1	1		
1	1					1	1			1	1	1	
	1	1				1	1			1	1	1	
		1	1			1	1			1	1	1	
		1	1		1	1				1	1	1	
		1	1	1	1					1	1	1	
	1	1		1	1					1	1	1	
1	1			1	1					1	1	1	
1	1			1	1						1	1	1

Table 6

A 6-dimensional complex with 14 vertices of large diameter

APPENDIX 3

FACE DECOMPOSABILITY

We outline in this appendix the dual notion of k-decomposability, namely "face decomposability." This involves the removal of a face and all of its subfaces from a polyhedron in such a way that the removed faces form a "pure" and "face decomposable" set and the remaining faces also form a "pure" and "face decomposable" set. It turns out that there is a class of convex sets whose face structures reflect this operation. They are not themselves polyhedra, but what we will call "generalized polyhedra." The remainder of the appendix will be spent making precise the connection between generalized polyhedra and k-decomposability. We reiterate that this appendix is merely a sketch, and hence the lemmas and many of the assertions, all of an elementary nature, will be stated without proof.

Definition A3.1: A generalized polyhedron is any set of the form $P = \{x \in \mathbb{R}^d \mid Ax \; \alpha \; b\}$, where A is an $n \times d$ matrix, b an n-vector, and $\alpha \in \{\leq, <, =\}^n$. The base polyhedron for P is the polyhedron $\overset{\circ}{P} = \{x \in \mathbb{R}^d \mid Ax \leq b\}$. (Note that $\overset{\circ}{P}$ depends not only on P, but also on the representation of P by A, b, and α). Given any $i \in \{1,\ldots,n\}$ for which α_i is $\leq$, define the deletion and link generalized polyhedron, respectively

$$P^{<}(i) = \{x \in \mathbb{R}^d \mid Ax \; \alpha^{<} \; b\}, \quad \alpha_j^{<} = \begin{cases} \alpha_j & j \neq i \\ < & j = i \end{cases}$$

$$P^{=}(i) = \{x \in \mathbb{R}^d | Ax \ \alpha^{=} \ b\}, \quad \alpha_j^{=} = \begin{cases} \alpha_j & j \neq i \\ = & j = i \end{cases} .$$

A generalized polyhedron P is clearly a convex set, and hence has a well defined face structure. We define vertices, edges, and diameters just as in Chapter 2. $P^{<}(i_0)$ and $P^{=}(i_0)$ are likewise generalized polyhedra, and their face structure is related to that of P as follows:

Lemma A3.2: Let P, P^0, $P^{<}(i)$, $P^{=}(i)$ be defined as above. Then for any face F of P, $P^{<}(i)$, or $P^{=}(i)$, $\overline{F}$ = the topological closure of F is a face of $\overset{o}{P}$. Further, if F is a face of P, then $\overline{F} = E_1$ for some face E_1 in $P^{<}(i)$ iff $F \not\subseteq P^{=}(i)$, and $\overline{F} = \overline{E}_2$ for some face E_2 in $P^{=}(i)$ iff $F \subseteq P^{=}(i)$.

Throughout this appendix, we will always take the base polyhedron $\overset{o}{P}$ to be a simple d-polyhedron. Hence we can define the dual complex Σ_P^* to P exactly as we did for simple polyhedra, that is, for $f_1, \ldots, f_m$ the facets ((d-1)-faces) of P, Σ_P^* is defined on set $E = \{v_1, \ldots, v_m\}$ by

$$\Sigma_P^* = \{v_{i_1} \ldots v_{i_k} | f_{i_1} \cap \ldots \cap f_{i_k} \text{ is a non-empty face of } P\}.$$

Then Σ_P^* is a simplicial complex, since if $f_{i_1}, \ldots, f_{i_k}$ intersect in a face of P, then any subcollection of $f_{i_1}, \ldots, f_{i_k}$ also

intersect in a face of P. Further, any face of P can be represented as the intersection of those facets containing that face. With the help of Lemma A3.2 we have

<u>Lemma A3.3</u>: Let P, f_i, and Σ_P^* be defined as above, and $i \in \{1,\ldots,n\}$ be chosen so that α_i is $\leq$ and the hyperplane $\{x \mid \langle A_i, x\rangle = b\}$ intersects P in a non-empty face F. Let $F = f_{i_1} \cap \ldots \cap f_{i_k}$. Then

1) $\Sigma^*_{P^<(i)} = \Sigma_P^* \setminus v_{i_1} \ldots v_{i_k}$, and

2) $\Sigma^*_{P^=(i)} = \ell k_{\Sigma_P^*}\, v_{i_1} \ldots v_{i_k}$.

We now have convex sets whose face structures correspond exactly to deletion and link in the dual simplicial complexes, and hence are in a position to define the properties of generalized polyhedron which correspond to a k-decomposition of the dual complexes. We will use Definition 3, and so to complete the connection, we need to define the properties dual to the initial and dimensionality conditions in Definition 3.

<u>Lemma A3.4</u>: Let P, $P^<(i)$, $P^=(i)$, $F = f_{i_1} \cap \ldots \cap f_{i_k}$ be defined as in Lemma A3.3. Then

1) Σ_P^* is a (d-1)-simplex iff every facet (equivalently, every face) of P contains the same vertex

2) $\Sigma^*_{P^<(i)}$ and $\Sigma^*_{P^=(i)}$ are (d-1)-dimensional and (d-k-1)-dimensional, respectively, iff they contain at least one vertex.

Now keeping Lemma A3.3 and Lemma A3.4 in mind, we define the concept of face decomposability:

Definition A3.5: A generalized polyhedron P is face decomposable if either every facet (equivalently, every face) of P contains the same vertex, or there exists an $i \in \{1,\ldots,n\}$, with α_i being $\leq$ and the face $F = P \cap \{x \mid \langle A_i, x\rangle = b_i\}$ being non-empty, such that

1) $P^<(i)$ contains a vertex and is face decomposable, and

2) $P^=(i)$ contains a vertex and is face decomposable.

P is called k-face decomposable if, in addition, $\dim F \geq k$ and $P^<(i)$, $P^=(i)$ *are both k-face decomposable.*

We have the immediate corollary to Lemma A3.3 and Lemma A3.4:

Proposition A3.6: If P is k-face decomposable then Σ^*_P is (d-k-1)-decomposable.

Further if it should happen that every face of P of dimension at least k is supported by a hyperplane corresponding to a row of $Ax \leq b$, then the implication in Proposition A3.6 is actually an equivalence.

The special case corresponding to vertex decomposability can be derived from the above statements as follows, noting that an essential inequality to the base polyhedron corresponds to a facet of the generalized polyhedron.

Proposition A3.6: Let P be a generalized polyhedron such that each row of the inequality $Ax \leq b$ is essential to the definition of the base polyhedron $\overset{\circ}{P}$. Then P is face decomposable iff Σ_P is vertex decomposable.

Corollary A3.7: If P as defined in Proposition A3.7 is face decomposable, then P satisfies the Hirsch Conjecture (for polyhedra).

We end this section with the dual notion to weak k-decomposability, namely "weak face decomposability." We will use definition 2^W, and hence need to establish the dual condition to pure (d-1)-dimensionality.

Lemma A3.8: Let P be a generalized polyhedron with a simple d-dimensional base polyhedron. Then Σ_P^* is pure (d-1)-dimensional iff every facet of P contains at least one vertex.

The definition of weak face decomposability is accordingly:

Definition A3.8: A generalized polyhedron P is weakly face decomposable if either every facet (equivalently, face) of P contains the same vertex, or there exists an $i \in \{1,\ldots,n\}$, with α_i being $\leq$ and the face $F = P \cap \{x | \langle A_i, x \rangle = b_i\}$ being non-empty, such that $P^<(i)$ has each of its facets containing a vertex and $P^<(i)$ is weakly face decomposable. P is called weakly k-face decomposable if, in addition, $\dim F \geq k$ and $P^<(i)$ is weakly k-face decomposable.

It follows:

Proposition A3.9: If P is weakly k-face decomposable, then Σ_P^* is weakly (d-k-1)-decomposable.

Again, if it should happen that every face of P of dimension at least k is supported by a row of $Ax \leq b$, then the implication is actually an equivalence.

Finally, as in Corollary 3.4.4:

Corollary A3.9: (Weakly) k-face decomposable generalized polyhedra have (polyhedral) diameter bounded above by a polynomial in n of degree $d-k$.

APPENDIX 4

SHELLABILITY AND RELIABILITY OF BOOLEAN SYSTEMS

Let f be a <u>monotone Boolean function</u>, i.e., a function mapping subsets of the set $E = \{1,\ldots,n\}$ into $\{0,1\}$ which has the property that for each pair S,T of subsets of E with $S \subseteq T$, $f(S) = 1$ implies that $f(T) = 1$. Suppose further that $p = (p_1,\ldots,p_n)$ is an assignment of probabilities to the set E so that each element i has independent probability p_i of appearing in a subset of E. Call the pair (f,p) a <u>Boolean system</u>. We are concerned with calculating the probability $P(f,p)$ that f will take on the value 1 over all subsets of E. Some examples:

<u>Example A4.1</u>: Let G be a connected graph, E the edges of G, and f the function which takes on the value 1 on $S \subseteq E$ whenever S contains a spanning tree. Then $P(f,p)$ is the probability that each pair of vertices G can be connected by a path of "operative" edges, given independent probabilities p_i of edge i being "operative," $i = 1,\ldots,n$.

<u>Example A4.2</u>: Let $Q = \{x \in \mathbb{R}^d_+ | Ax = b\}$ be a polyhedron, $E = \{1,\ldots,d\}$, and f the function which takes on the value 1 on $S \subseteq E$ whenever there exists a point in Q whose only non-zero components are in S. Then $P(f,p)$ is the probability that the system $Ax = b$, $x \geq 0$ has a solution, given independent probabilities p_i of component i being non-zero, $i = 1,\ldots,d$.

<u>Example A4.3</u>: Let $E = \{1,\dots,n\}$, $w_i \geq 0$ the "weight" given to each "player" i in E, q the "quota", and f the function which takes on the value 1 on $S \subseteq E$ whenever $\sum_{i \in S} w_i \geq q$. Then $P(f,p)$ is the probability that a "motion" will pass given the independent likelihood p_i that player i will vote for the motion.

Calculation of $P(f,p)$ is rather tedious. The straightforward expression for this number is

$$P(f,p) = \sum_{S \subseteq E} f(S) \prod_{i \in S} p_i \prod_{i \notin S} (1-p_i)$$

which has 2^n terms. Ball [4] has simplified this expression considerably, using the notion of an "elementary partition". Consider the set $\Gamma = \{S \subseteq E \mid f(S) = 1\}$. An <u>elementary partition</u> of Γ is a partition of the members of Γ into sets of the form $(\tau,\sigma) = \{S \mid \tau \subseteq S \subseteq \sigma\}$ where $\tau \subseteq \sigma$ and are members of Γ. (Note that $\tau \in \Gamma$ implies that every S containing τ must also be in Γ, and so $(\tau,\sigma) \subseteq \Gamma$.) At least one elementary partition of Γ always exists, namely that which has $\tau = \sigma$ and σ running through all of the members of Γ. Now given an elementary partition $(\tau_1,\sigma_1),\dots,(\tau_m,\sigma_m)$ of Γ we simply observe that

$$1 - P(f,p) = \Pr\{S \in \Gamma\}$$

$$= \sum_{j=1}^{m} \Pr\{S \in (\tau_j,\sigma_j)\}$$

$$= \sum_{j=1}^{m} \Pr\{i \in S \text{ for each } i \in \tau_j \text{ and } i \notin S \text{ for each } i \notin \sigma_j\}$$

$$= \sum_{j=1}^{m} \prod_{i \in \tau_j} p_i \prod_{i \notin \sigma_j} (1-p_i)$$

and this expression has m terms, where m is the number of sets in the partition. It is clear that m is at least the number of minimal members of Γ, and hence the number of terms in the above expression can be minimized if we could find an elementary partition $(\tau_1,\sigma_1),\ldots,(\tau_m,\sigma_m)$ where $\tau_1,\ldots,\tau_m$ are exactly the minimal members of Γ.

It turns out to be easier, for our purposes, to consider the complementary collection $\Sigma = \{E\backslash S \mid S \in \Gamma\}$. Σ is a simplicial complex (except for the case $f \equiv 1$), since $E\backslash S \in \Sigma$ and $E\backslash T \subseteq E\backslash S$ imply that $E\backslash T \in \Sigma$. Further, if $(\tau_1,\sigma_1),\ldots,(\tau_m,\sigma_m)$ is an elementary partition of Σ, then $(E\backslash\sigma_1, E\backslash\tau_1),\ldots,(E\backslash\sigma_m, E\backslash\tau_m)$ is an elementary partition of Γ, and if $\sigma_1,\ldots,\sigma_m$ are maximal simplices in Σ, then $E\backslash\sigma_1,\ldots,E\backslash\sigma_m$ are minimal elements of Γ. Call such a partition of Σ a <u>full</u> (elementary) <u>partition</u> and call Σ <u>full partitionable</u>.

We present here a large and interesting class of full partitionable complexes.

<u>Definition A4.4</u>: A (not necessarily pure) simplicial complex Σ is <u>shellable</u> if the maximal simplices of Σ can be ordered $\sigma_1,\ldots,\sigma_m$

so that, for $i > 1$, $\bar{\sigma}_i \cap (\bigcup_{j=1}^{i-1} \bar{\sigma}_j)$ is a pure $(|\bar{\sigma}_i|-2)$-dimensional complex.

Note that this is a generalization of the definition of shellable given in Section 2.5.

Theorem A4.5: Shellable complexes are full partitionable.

Proof: Let Σ be shellable, with shelling order $\sigma_1,\ldots,\sigma_m$. For $i = 1,\ldots,m$ define

$$\tau_i = \cup \{\sigma_i \setminus \mu \mid \mu \text{ a } (|\sigma_i|-2)\text{-simplex in } \bar{\sigma}_i \cap (\bigcup_{j=1}^{i-1} \bar{\sigma}_j)\}$$

(so that $\tau_1 = \emptyset$). Then $\tau_i \subseteq \sigma_i$, and hence are both simplices in Σ. Further, $\rho \in (\tau_i,\sigma_i)$ iff $\rho \subseteq \sigma_i$ and ρ contains $\sigma_i \setminus \mu$ for each $(|\sigma_i|-2)$-simplex μ in $\bar{\sigma}_i \cap (\bigcup_{j=1}^{i-1} \bar{\sigma}_j)$ iff ρ is contained in σ_i but in no σ_j $j < i$. But this means that $(\tau_1,\sigma_1),\ldots,(\tau_m,\sigma_m)$ partition the simplices of Σ, and hence Σ is full partitionable.

Notice that if Σ is pure dimensional, then the τ_i chosen in the proof of Theorem A4.5 are precisely the shedding simplices for a d-decomposition of Σ as chosen in Theorem 3.3.

Corollary A4.6: Independent sets of matroids, broken circuit complexes, dual complexes to simple polyhedra, and distributive lattice complexes are full partitionable.

In particular, Examples A4.1 and A4.2 (for simple polyhedra) are full partitionable, since the Σ associated with A4.1 is the dual matroid of the polygon matroid in the graph G (see [37] §2.2) and the Σ associated with A4.2 is the dual complex to the polyhedron.

<u>Proposition A4.7</u>: Let $w_1 \geq w_2 \dots \geq w_n \leq r$ non-negative numbers, and define the simplicial complex Σ on set $E = \{v_1, \dots, v_n\}$ by

$$\Sigma = \{v_{i_1} \dots v_{i_k} \mid \sum_{j=1}^{k} w_{i_j} \leq r\}.$$

Then Σ is shellable.

<u>Proof</u>: Let $\sigma_1, \dots, \sigma_m$ be a lexicographic ordering of the maximal simplices of Σ, that is, $\sigma_i = v_{i_1}, \dots, v_{i_p}$, $i_1 < \dots < i_p$, comes before $\sigma_j = v_{j_1} \dots v_{j_q}$, $j_1 < \dots < j_q$, if and only if $i_1 = j_1, \dots, i_k = j_k$, $i_{k+1} < j_{k+1}$ for some k. Note that this is well defined, since the non-existence of either i_{k+1} or j_{k+1} implies that one of σ_i, σ_j is contained in the other, and it can easily be checked that the ordering is transitive. We prove that $\sigma_m, \dots, \sigma_1$ is a shelling of Σ. For suppose τ is a maximal element of $\bar{\sigma}_i \cap (\bigcup_{j=i+1}^{m} \bar{\sigma}_j)$. Then $\tau = \sigma_i \cap \sigma_j$ for some $j > i$, so that if $\sigma_i = v_{i_1} \dots v_{i_p}$, $i_1 < \dots < i_p$, and $\sigma_j = v_{j_1} \dots v_{j_q}$, $j_1 < \dots < j_q$, then for some k, $i_1 = j_1, \dots, i_k = j_k$, $i_{k+1} < j_{k+1}$. Thus $v_{i_{k+1}} \notin \tau$. Suppose a second $v_{i_s} \in \sigma_i \backslash \tau$, so that $s > k+1$. Consider the set $\tau \cup \{v_{i_s}\}$. We have

$$\sum_{v_t \in \tau \cup \{v_{i_s}\}} w_t \leq \sum_{v_t \in \sigma_i} w_t \leq r, \quad \text{and}$$

$$\sum_{v_t \in \tau \cup \{v_t | t > i_{k+1}\}} w_t \geq \sum_{v_t \in \sigma_j \cup \{v_{i_s}\}} w_t > r,$$

hence there must be a maximal proper set σ' in $\tau \cup \{v_t | t > i_{k+1}\}$ containing $\tau \cup \{v_{i_s}\}$ with the property that $\sum_{v_t \in \sigma'} w_t \leq r$. But then σ' is a maximal element in Σ, since any element v_t, $t \leq i_{k+1}$, has at least as large w_t as any v_t, $t > i_{k+1}$. Further, σ_i comes before σ' in the lexicographic ordering, since $\sigma' = v_{i_1} \ldots v_{i_k} v_{\ell_{k+1}} \ldots$, $i_1 < \ldots < i_k < \ell_{k+1} < \ldots$, with $\ell_{k+1} > i_{k+1}$. So we have a contradiction between $\sigma_i \cap \sigma' \supseteq \tau \cup \{v_{i_s}\}$ and τ being a maximal simplex in $\bar{\sigma}_i \cap (\bigcup_{j=i+1}^{m} \bar{\sigma}_j)$. Therefore no such i_s exists, and so $|\tau| = |\sigma_i| - 1$, proving that $\sigma_m, \ldots, \sigma_1$ is a shelling.

<u>Corollary A4.7</u>: The complex defined in Proposition A4.6 is full partitionable.

In particular, the complex corresponding to Example A4.3 is full partitionable, since the corresponding Σ is of the form described in Proposition A4.7, with $r = (\sum_{i=1}^{n} w_i) - q$.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: Technical Report No. 354	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): DECOMPOSITIONS, SHELLINGS, AND DIAMETERS OF SIMPLICIAL COMPLEXES AND CONVEX POLYHEDRA	5. TYPE OF REPORT & PERIOD COVERED: Technical Report
	6. PERFORMING ORG. REPORT NUMBER: T.R. No. 354
7. AUTHOR(s): John Scott Provan	8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0678
9. PERFORMING ORGANIZATION NAME AND ADDRESS: School of Operations Research and Industrial Engineering, Cornell University, Upson Hall Ithaca, New York 14853	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Operations Research Program Office of Naval Research Arlington, VA 22217	12. REPORT DATE: Aug. 1977
	13. NUMBER OF PAGES: 123
11. CONTROLLING OFFICE NAME AND ADDRESS	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Polyhedron	Decomposition	Piecewise Linear Topology
Simplicial Complex	Hirsch Conjecture	
Diameters	Dual Complex	
Shellability	Homology	

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Simplicial complexes are studied in their role dual to convex polyhedra, in particular, with respect to shellability and diameters. The concept of k-decomposability is presented which both insures shellability and limits the diameter of a simplicial complex. k-decomposability is studied in relation to standard techniques used in polyhedral theory, and with respect to topological properties of simplicial complexes.

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